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A MARKOVIAN ANALYSIS OF THE LIFE OF NEWSPAPER SUBSCRIPTIONS*

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This paper presents a case study in the application of some statistical theory, and a mathematical model, to an actual business problem. The problem was to determine the pattern of mortality of subscriptions to a daily newspaper, and thereby to ascertain the average life of such subscriptions. The method of solution, herewith indicated, involved the collection of data from a probability sample of subscribers to the newspaper, and use of these data to construct a Markov chain.

The study that we describe in this paper followed a probability sample of over 2000 subscribers to the Detroit Evening News during a one year period to ascertain rates of cancellation among various groups of subscribers. The data so obtained served as input to a mathematical model of the system of subscription and cancellation. The particular model that we adopted is a Markov chain, which we shall describe later. Based on the data and the model, we constructed a mortality table for subscriptions and calculated the average life of subscriptions in various classes.

The figures in this paper have been altered to preserve the confidentiality of the actual data obtained by the study.[1] The figures we present merely serve the purpose of illustrating the theory that we employed. The conclusions drawn in the paper are consistent with the figures presented, not necessarily with the real-world situation.

The original motivation for the study was a tax-case that the Detroit News was involved in. The management of the News had, a few years earlier, purchased subscriptions from another Detroit newspaper when the later ceased publication. To provide a basis for the depreciation of such subscriptions, management required the average life of subscriptions to the News. Although the tax case was settled meanwhile on a different basis, the results of the study turned out to be useful from a business standpoint. Here for the first time, a newspaper had figures not only on the mortality of subscriptions, but in addition, as by-products, several other items of information that were helpful to management. For example, these items of information included the numbers of subscribers that carried accident insurance under the auspices of the News, together with the distribution of the number of policies that a subscriber car-

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[1] It is a principle of our work that the statistical methods that we develop in an engagement are ours to publish, but that the data are the property of the client.

ried, the length of life of various classes of subscription (e.g., old subscriptions, new ones, insured, uninsured), and the relative frequency of reasons that subscribers gave for cancelling their subscriptions.

General Description of the Mathematical Model

The first step in the statistical study of subscriptions was to develop a mathematical model to describe the pattern of life and death of subscriptions. It was instructive and fruitful to view this pattern as a Markov process, wherein at any given time a subscriber could be classified as being in some "state." The basic idea underlying this approach is that the particular state that a subscriber is in at any given time determines his probability of continuing or cancelling his subscription.

The next step was accordingly to identify various groups of subscribers that were likely in the judgment of all concerned to exhibit different rates of termination. It was decided to group subscribers on two criteria:

1. According to whether the subscription was relatively recent. In this connexion, we were able to classify subscriptions as being (a) less than 1 year old, (b) from 1 to less than 2 years old, and (c) 2 or more years old. This classification was accomplished by tracing names of subscribers that fell into the sample to ABC-lists of subscribers for past years.[2]

2. According to whether the subscriber carried News-sponsored accident insurance. Only subscribers to the News are permitted to carry such a policy. Hence, subscribers that carry insurance may be more reluctant than others to give up their subscriptions, lest they lose their inusrance as well.

Thus, for purposes of this study we defined 7 possible categories of subscription, as follows:

U-0. Subscriber uninsured, subscription less than 1 year old;
U-1. Subscriber uninsured, subscription from 1 to less than 2 years old;
U-2. Subscriber uninsured, subscription 2 or more years old;
I-0. Subscriber insured, subscription less than 1 year old;
I-1. Subscriber insured, subscription from 1 to less than 2 years old;
I-2. Subscriber insured, subscription 2 or more years old;
C. Súbscription cancelled.

The nature of our model is that each subscriber to the News is in one of these 7 possible states at a given time. That is, the subscriber may be insured, or he may not be insured, and in either case he may have subscribed for less than 1 year, for less than 2 years, or for 2 years or more. Furthermore, a subscriber may, from one year to the next, shift from one state to any one of 2 or 3 other possible states. For example, imagine a subscriber in state *I*-1. The following year that subscriber may have shifted to state *I*-2 (which means that he has continued his subscription and his insurance), to state *U*-2 (which means that

[2] An ABC-list is a sworn list of paid subscribers, sent in annually in March by newspapers and magazines to the Audit Bureau of Circulation.

he has continued his subscription but not his insurance), or to state C (which means that he has terminated his subscription and automatically his insurance). Similarly, a subscriber in state U-2 may shift, by the next year, to I-2 or to C, or he may remain in U-2. Other transitions, by their nature, are not possible.

Last, we note that once a subscriber moves to state C (cancellation), no further transitions are possible. This is an absorbing state; hence, our model is an absorbing Markov chain [3, pp. 43–68].

Our model is such that after 2 years, a subscriber will, at any given time, be in either state U-2 or I-2, if he has not already been absorbed in state C. Thus, the model assumes that after an initial "wear-in" period of 2 years, rates of cancellation are relatively stable, except to the degree that they depend on the insurance status of the subscriber. After the survey of subscribers was completed we did, in fact, examine some results to ascertain if the rate of cancellation does vary with age of subscription beyond the 2-year period. The data did not suggest significant trends. Hence, there appeared to be no need to modify the model, though such modification would be straightforward from a mathematical point of view.

The Survey of Subscribers

Once the mathematical model had been formulated, the next step in the study was to estimate, by a statistical survey of subscribers, a matrix of transition probabilities to reflect the relative frequencies with which subscribers in the various states each year move to other states.

As the introduction indicated, this statistical survey involved tracing a probability sample of subscribers to the Detroit News for a year, March 1965 to March 1966, to ascertain rates of cancellation among the various groups of subscribers.

Allocation of responsibilities

The sampling procedure specified that the frame, the definition of a subscriber, the definition of a cancellation, and procedures by which to match names of subscribers on past ABC-Lists, would be the responsibility of the Detroit News, as these problems are substantive. The Detroit News had also the responsibility to carry out with care and accuracy the statistical procedures specified by the statisticians. The statistical procedures included the design and an evaluation of the statistical reliability of the results.

What administrative action to take on the basis of the results was entirely up to the Detroit News.

Definition of a subscriber

The definition of a subscriber that the management of the News adopted for purposes of this study is, in essence, "any person, group of persons, or business establishment that has the newspaper delivered on a regular basis, by paper paper boy, by route truck, or by mail." It will be observed that this definition excludes sales by newsstands.

The frame

The frame was by definition the list of subscribers that if studied in its entirety would provide the information required. Management of the News elected to use as a frame the ABC-list of subscribers to the News as of 16 March 1965, plus the mailed subscriptions. The frame consisted of several thousand pages. Each page had 35 lines, numbered 1 to 35 by the printer. The sampling unit for the study was a line.

A line on the ABC-list shows name and address and a few other items of information for a subscriber that has his newspaper delivered by a paper-boy or by a route-truck in an outlying area. The ABC-list also includes the few subscribers that receive the paper by mail. It indicates whether the subscriber takes the daily paper, the Sunday paper, or both, the name of the paper-boy or route-driver, and indication of the route and district.

Design of the probability sample

The sample was selected by random numbers from the frame. The whole procedure met the requirements of a probability sample, which is one for which the selection of the sample and calculation of estimates permit use of the theory of probability to calculate, from the results themselves, the margin of uncertainty that one may reasonably ascribe to small independent accidental variations, including the effect of sampling.

Instructions to the News, with respect to the selection of the sample, required that they number the pages of the ABC-list 1, 2, 3, and onward so that every page had a serial number.

Within each zone of 100 consecutive pages, random numbers drew one page into each of the 5 subsamples. For each page so selected, random numbers drew 2 of the 35 lines into the sample. The random selections were furnished by the authors in the form of a sampling table, and transmitted to the News. It was the responsibility of the News to draw off, from the ABC-list of March 1965, names and addresses of sample subscribers on the lines designated by the sampling table. A line with no name was a blank: no substitution was permitted.

Tabulation of results for each of the 5 subsamples separately, and for all 5 subsamples combined, facilitated computation of standard errors.

Definition of a cancellation

For purposes of the study the management of the News defined a cancellation as follows:

> The termination of regular delivery; provided that any temporary interruption of regular delivery to a person, such as for vacations or other temporary absence of the person from the address to which regular delivery is made, shall not be deemed a termination of regular delivery.
>
> A subscription shall be deemed to be a continuing subscription so long as regular delivery is maintained to any member of a single household continuing to reside as a part thereof. If any member removes permanently

from such household and if regular delivery is continued to any remaining member of such household, the subscription is not to be deemed terminated.

For purposes of the above definition, an interruption of regular delivery for an indefinite period expected to exceed six months, shall not be deemed to be a temporary interruption of regular delivery.

Collection of data

The responsibility for the collection of data and for the tabulation of results rested with the management of the News. The sampling procedure asked the News to record on appropriate forms the names and addresses of the subscribers selected according to the sampling table, and then to follow all the names in the sample throughout the year to ascertain which subscribers terminated their subscriptions.

The management of the News developed procedures by which to watch subscribers in the sample, that is, procedures by which to trace every subscriber that moved during the year, and to ascertain whether he continued his subscriptions at a new address. These procedures involved continual verification of files in district offices, with special instructions to supervisory personnel to verify immediately with the subscriber any notice of cancellation.

In addition to specifying a watch for cancellations, the procedures also required the News (a) to follow subscribers insured as of March 1965 to ascertain which ones continued their subscriptions but dropped their insurance during the year, and (b) to report subscribers uninsured as of March 1965 that took out insurance during the year.

Further, the procedures requested the News to collect information on the ages of subscriptions by attempting to locate the names of subscribers in the sample (drawn from the March 1965 ABC-list) on the ABC-list of March 1964 and on the ABC-list of March 1963. In our subsequent analysis of the data, we have treated subscribers that the News located on both prior ABC-lists as subscribers for "2 years or more;" subscribers that the News located only on the March 1964 ABC-list as subscribers for "1 year to less than 2 years;" and subscribers that the News could not locate on the ABC-list of March 1964 as subscribers for "less than 1 year."

Tabulation of data

The procedures requested the News to compile the following information:

1. The number of subcribers and the number of terminations observed in the sample, classified by insurance status as of March 1965 (insured or uninsured), by category of match with prior ABC-lists (matched to both the 1964 and 1963 lists, matched to the 1964 list only, and not matched even to the 1964 list), by subsample.

2. The number of subscribers in the sample that were insured as of March 1965 and had continued their subscriptions through March 1966, but had dropped their insurance as of March 1966, by category of match with prior ABC-lists, and by subsample.

TABLE 1

The transition matrix

Present state	Subsequent state						
	U-0	*U*-1	*U*-2	*I*-0	*I*-1	*I*-2	*C*
U-0	.0000	.7397	.0000	.0000	.0152	.0000	.2451
U-1	.0000	.0000	.8404	.0000	.0000	.0142	.1454
U-2	.0000	.0000	.9278	.0000	.0000	.0134	.0588
I-0	.0000	.0000	.0000	.0000	.8571	.0000	.1429
I-1	.0000	.0000	.0000	.0000	.0000	.8333	.1667
I-2	.0000	.0000	.0433	.0000	.0000	.9195	.0372
C	.0000	.0000	.0000	.0000	.0000	.0000	1.0000

3. The number of subscribers in the sample that were not insured as of March 1965, had continued their subscriptions through March 1966, and had become insured as of March 1966, by category of match with prior ABC-lists, by subsample.

These tabulations provided probabilities such as those shown in the transition matrix, Table 1. (As noted earlier, the figures presented here are altered.) For example, the probability of .0372 in the 6th row and 7th column of the matrix could have been derived from the fact that the survey found 484 insured subscribers that had been subscribing for 2 years or more, and that of these, 18 cancelled their subscriptions during the year of the survey. Thus, 18/484 = .0372 would represent the relative frequency of subscribers in state *I*-2 (subscriber insured, subscription 2 or more years old) that, in a year, moved to state *C*.

Some Theory and Numerical Results

The rationale underlying the application of a mathematical model is that once one describes a business operation mathematically, he may answer a number of pertinent questions about that operation on the basis of theory. In the present case, the pertinent questions about the operation centered on the mortality pattern of subscriptions to the News, and were answered by various mathematical manipulations of the transition matrix. The mortality table, presented as Table 2, which was derived from Table 1, and hence based on altered figures, provides answers to a number of important questions about termination of subscriptions.

The appropriate theory for Markov chains in general, and for an absorbing Markov chain in particular, is covered in [3]. Additional results, and a somewhat different business application, appear in [1]. Here we show the application of some of that theory to our problem.

The starting vector

The particular set of questions that we deal with here refer to the mortality pattern of subscribers, such as the group that subscribed to the News in March

TABLE 2

Mortality table for subscriptions to a newspaper

Year x	Probability of cancellation			Probability of continuation through year x			Expected life after x years
	In year x	Within x years	Hazard rate	Uninsured	Insured	Total	
0				0.8427	0.1573	1.0000	17.42
1	0.1066	0.1066	0.1066	0.7379	0.1554	0.8934	18.38
2	0.0633	0.1699	0.0708	0.6777	0.1524	0.8301	18.70
3	0.0455	0.2154	0.0548	0.6353	0.1492	0.7846	18.73
4	0.0429	0.2584	0.0547	0.5959	0.1457	0.7416	18.76
5	0.0405	0.2988	0.0546	0.5592	0.1419	0.7012	18.78
6	0.0382	0.3370	0.0544	0.5250	0.1380	0.6630	18.80
7	0.0360	0.3730	0.0543	0.4931	0.1339	0.6270	18.83
8	0.0340	0.4070	0.0542	0.4633	0.1297	0.5930	18.85
9	0.0321	0.4391	0.0541	0.4355	0.1255	0.5609	18.87
10	0.0303	0.4693	0.0540	0.4095	0.1212	0.5307	18.89
11	0.0286	0.4979	0.0539	0.3852	0.1169	0.5021	18.91
12	0.0270	0.5249	0.0538	0.3624	0.1126	0.4751	18.92
13	0.0255	0.5504	0.0537	0.3411	0.1084	0.4496	18.94
14	0.0241	0.5745	0.0536	0.3212	0.1043	0.4255	18.96
15	0.0228	0.5973	0.0535	0.3025	0.1002	0.4027	18.97
16	0.0215	0.6188	0.0534	0.2850	0.0961	0.3812	18.99
17	0.0203	0.6392	0.0534	0.2686	0.0922	0.3608	19.00
18	0.0192	0.6584	0.0533	0.2532	0.0884	0.3416	19.01
19	0.0182	0.6766	0.0532	0.2388	0.0847	0.3234	19.03
20	0.0172	0.6938	0.0532	0.2252	0.0810	0.3062	19.04
21	0.0163	0.7100	0.0531	0.2125	0.0775	0.2900	19.05
22	0.0154	0.7254	0.0530	0.2005	0.0741	0.2746	19.06
23	0.0145	0.7399	0.0530	0.1892	0.0708	0.2601	19.07
24	0.0138	0.7537	0.0529	0.1786	0.0677	0.2463	19.08
25	0.0130	0.7667	0.0529	0.1687	0.0646	0.2333	19.09
Over 25	0.2333	1.0000	1.0000				

1965. This group includes a certain proportion of insured subscribers and uninsured subscribers, and in both of these classes, a certain proportion of those that had subscribed for less than 1 year, for less than 2 years, and for 2 years or more. We introduce a 7-component probability vector to describe the composition of this reference group. The components of this vector show the proportion of subscribers observed in the states U-0, U-1, U-2, I-0, I-1, I-2, C as of March 1965. The vector we shall use here for purposes of illustration is as follows:

$$(.2083, .1274, .5070, .0032, .0081, .1460, .0000).$$

Each component of the vector is the probability that a subscriber, selected at random at the time we begin to study the group, will be in a given state. The

2d component, for example, suggests that as we begin such study of the group .1274 of the subscribers are uninsured and hold subscriptions that are from 1 to 2 years old. On this interpretation of the starting vector, the resulting mortality table traces a subscriber, selected at random at some point in time, perhaps some time after he has commenced his subscription. This is in contrast to most mortality tables which trace mortality patterns from birth. Here we are interested in how an existing group will die off; hence the somewhat different concept.

Powers of the transition matrix

To write out certain pertinent formulas, we introduce the symbol P to represent the 7×7 transition matrix that appears in Table 1, and the symbol w to represent the 7-component row vector that provides our starting probabilities.

As is well known from the theory of Markov chains, a transition matrix raised to the power x gives the probabilities of shifting from one state to another in x steps. It follows that, in our problem, the last column of P^x provides probabilities of cancellation within x years, conditional on the starting states. Further, wP^x is a 7-component probability vector whose components are the probabilities that a subscriber will be in one of the 7 basic states of the Markov chain after x years. For example, take

$$\begin{aligned} wP &= (.0000,\ .1541,\ .5838,\ .0000,\ .0059,\ .1496,\ .1066) \\ wP^2 &= (.0000,\ .0000,\ .6777,\ .0000,\ .0000,\ .1524,\ .1699) \\ wP^3 &= (.0000,\ .0000,\ .6353,\ .0000,\ .0000,\ .1492,\ .2154) \\ &\ \vdots \end{aligned}$$

The entry at the extreme right in wP^x is the probability that a subscription will be cancelled within x years, which may be written

$$F(x) \equiv P(\tilde{x} \leqq x) = wP^x v$$

where we introduce the symbol $\tilde{x}$ for the random variable "time-to-cancellation" and the symbol v for a 7-component column vector with its first 6 components equal to 0 and the 7th equal to 1.

The sum of the first 6 components of wP^x gives the probability that a subscription will continue for at least x years; the sum of the first 3 components alone gives the probability that after x years the subscriber will continue his subscription and be uninsured, and the sum of the second 3 components gives the probability that he will continue his subscription after x years and be insured.

The conditional probability that a random subscriber will be initially insured is, according to our illustrative figures, .1573. After 25 years this probability has increased to $.0646/.2333 = .2769$. This probability is, in fact, monotonically increasing to a limit which depends on the transition probabilities and is independent of the starting probabilities. This limit can readily be shown to be .3187 for our numerical illustration.

The probability $f(x)$ that a subscription will be cancelled in exactly the x-th year is readily calculated by differences: thus

$$f(x) \equiv P(\tilde{x} = x) = F(x) - F(x\text{-}1)$$

where by definition $F(0) = 0$. It is also interesting to examine the conditional probability of cancellation in the x-th year; that is, the hazard rate as defined by the relation

$$h(x) = f(x)/[1 - F(x\text{-}1)].$$

As can be seen from the mortality table, the hazard rates calculated from Table 1 are in this application monotonically decreasing over time. Initially this reflects the fact that new subscriptions have a relatively high mortality rate. Subsequently, over time, a higher proportion of the surviving subscribers carry insurance, with a lower risk of cancellation. The transition matrix shows, directly, that after a subscription is more than 2 years old, the hazard rate for subscribers with insurance is .0372, and that the hazard rate for subscribers without insurance is .0588. As noted above, the proportion of insured subscribers among survivors tends to .3187, and the proportion of uninsured subscribers tends to .6813. It follows that

$$\lim_{x\to\infty} h(x) = .3187 \times .0372 + .6813 \times .0588 = .0519.$$

The fundamental matrix

It is also instructive, in studying this system, to look at the so-called fundamental matrix of the absorbing Markov chain. This is a 6×6 matrix in the present problem, which we denote by the symbol N, and which is defined by the formula

$$N = I + Q + Q^2 + Q^3 + \cdots = (I - Q)^{-1}$$

I being a 6×6 identity matrix, and Q a 6×6 matrix equal to the transition matrix with the row and the column corresponding to the absorbing state C deleted. The calculated components of N for the present illustration are as follows:

$$N = \begin{bmatrix} 1.0000 & .7397 & 9.7567 & .0000 & .0152 & 1.9080 \\ .0000 & 1.0000 & 13.0484 & .0000 & .0000 & 2.3433 \\ .0000 & .0000 & 15.3861 & .0000 & .0000 & 2.5554 \\ .0000 & .0000 & 5.9177 & 1.0000 & .8571 & 9.8565 \\ .0000 & .0000 & 6.9040 & .0000 & 1.0000 & 11.4992 \\ .0000 & .0000 & 8.2848 & .0000 & .0000 & 13.7990 \end{bmatrix}$$

The element in row i and column j of N represents the expected number of years that a subscriber who started in the state i will be found in state j. If we let y be a 6-component column vector with all components equal to 1, Ny is also a 6-component column vector, wherein each component represents the expected life or time-to-cancellation of a subscription that starts out in the corresponding state (U-0, U-1, U-2, I-0, I-1, and I-2, respectively). We calculate the components of Ny to be as follows:

$$(Ny)^T = (13.42, 16.39, 17.94, 17.63, 19.40, 22.08).$$

If we further define u as a 6-component row vector equal to w with, however, the component of w which corresponds to the absorbing state C deleted, the expected life over all subscriptions is given by the formula

$$E(\bar{x}) = uNy.$$

Application of this formula shows that the average life of a subscription, for the assumed data, is 17.42 years.

The expected life of a particular kind of subscriber, such as an insured subscriber or a subscriber for less than 1 year, may be derived simply by setting components in u that refer to subscribers other than those of the type under consideration equal to 0, calling the new vector u', and computing

$$E(\bar{x}) = u'Ny/u'y.$$

For example, we find in our illustration that the average life of subscriptions for subscribers who start out insured is 21.86 years, while the average life of subscriptions for subscribers that start out uninsured is 16.59 years.

We may also compute conditional expectations of remaining life, given that a subscription has survived x years. For this purpose, one defines u_x as a 6-component row vector equal to wP_x with the last component deleted, and calculates

$$E(\bar{x} \mid x) = u_xNy/u_xy.$$

Some values of this conditional expectation are given in the mortality table (Table 2). Asymptotically, this conditional expectation tends to 19.26 years in our example. If one further requires a calculation of expected life for a particular type of subscriber, such as insured subscribers, he simply sets all other categories in u_x equal to 0 and computes $E(\bar{x} \mid x)$ from this same formula.

The proportion of time, over its lifetime, that a subscription will be found in a particular state may also be derived by defining y' as a 6-component column vector with components 1 or 0, depending on whether they refer to the particular kind of state, or not, and calculating

$$R = uNy'/uNy.$$

For example, in our illustration, the proportion of time that a subscriber is in in an insured state over the life of his subscription, is $4.1476/17.4175 = .2381$, and the proportion of time that he is in an uninsured state is $13.2699/17.4175 = .7219$.

Standard errors

Each of the foregoing numerical results is subject to some margin of uncertainty due to sampling and other small independent accidental variations. Our sample design which employed 5 independent subsamples, facilitated the computation of standard errors which, of course, objectively measure this type of uncertainty. The replicated design has also permitted us to investigate the existence of bias in the estimating formulas—that is, the possibility that the estimator, $\hat{\theta}$, that we adopt for a parameter, θ, is subject to bias, in the sense that $E(\hat{\theta})$ may differ from θ. Such investigation is to be recommended when the estimating formula is a complex function of the observations.

For example, based on the overall sample, the assumed data show the expected life is 17.42 years for all subscribers. Similar computations for each of 5 subsamples might show these results:

Subsample:	1	2	3	4	5
Estimate of $E(\bar{x})$:	17.48	17.26	12.70	20.59	23.94.

The standard error of the over-all estimate would then be 1.88 years with 4 degrees of freedom based on the usual sum-of-squares formula.

Detection of bias in the estimate of $E(\bar{x})$ may be examined by calculation of estimates of $E(\bar{x})$ based on the 5 combinations of 4 subsamples. The results for the present illustration are as follows:

Subsamples:	1-2-3-4	1-2-3-5	1-2-4-5	1-3-4-5	2-3-4-5
Estimate of $E(\bar{x})$:	17.55	17.48	19.27	16.90	16.34.

The average of these results is 17.51 years, which is statistically the same as the previous overall estimate of 17.42 years. There is thus no evidence of bias in the overall estimate. The same statement holds for the other pertinent results.

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