A POSSIBLE EXPLANATION OF PACKING

by

William Edwards Deming,

B.S.(E.E.), 1921, University of Wyoming,

M.S., 1924, University of Colorado,

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RECOGNIZED DIFFICULTIES WITH THE PACKING EFFECT

According to the electromagnetic theory of matter, packing should be accounted for by the negative mutual mass of electrons and protons. The evidence becomes more and more convincing with time that a helium atom must be made up of four protons and four electrons, that is, of four hydrogen atoms. This idea immediately encounters a small difficulty with atomic weights. The atomic weights of hydrogen and helium are known with little doubt to four figures to be 1.0078 and 4.000 respectively, whence the packing effect in helium must be $4 \times 1.0078 - 4.000 = 0.0312$. Harkins and Wilson,[1] by calculating the mutual electromagnetic momentum of two moving charges, found that if the distance between centers of an electron and a proton in the mucleus is about 400 times the radius of a proton, the required amount of negative mutual mass results. But this is obviously an impossible arrangement if the radius of an electron is anything like 1846 times that of a proton, as is suggested by the electromagnetic theory.

It is difficult to imagine any atomic model for helium which could give more packing than one with a half nucleus as shown in Figure 1. A and B are two protons diametrically adjacent to an electron. Yet this model yields only about one-tenth the requisite packing, as is easily proved.

Thus if

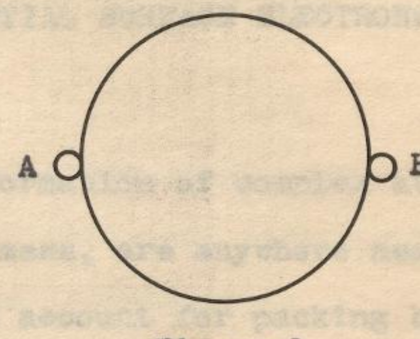


Figure 1

a = radius of electron,

A = radius of proton,

e = electronic charge in Heaviside units,

m = rest mass of an electron = $e^2/6\pi ac^2$,

M = rest mass of a proton = $m\ a/A$, the expression [2]

$$\frac{1}{6\pi c^2}\sum_{ij}\frac{e_i e_j}{r_{ij}}, \qquad i \neq j \tag{1}$$

for the space average mutual mass gives

$$2\,\frac{1}{6\pi c^2}\left\{-4\,\frac{e^2}{a+A} + 2\,\frac{e^2}{2a+2A}\right\} = \frac{2e^2}{6\pi\ a\ c^2}\cdot\frac{-3}{1+A/a} = -6.00\ m$$

for both halves, (using $a:A = 1846$). The difference 0.0312 between the atomic weights of $4H$ and He is equivalent to

$$0.0312 \times 1847m/1.0078 = 57.18\ m$$

since an atomic weight 1.0078 means each atom has the mass 1847 m. 6:57 is roughly 1:10 as stated above.

This result is independent of any <u>radial</u> law of density of charge in electron or proton, provided they remain spheres in the inertial system in which they are momentarily at rest, because homogeneous concentric layers of charge behave to external points as if the whole charge were concentrated at the center.

THE POSTULATE OF EQUIPOTENTIAL SURFACE ELECTRONS

If modern views on the formation of complex atoms, and the electromagnetic theory of mass, are anywhere near correct, there must be some way to account for packing by the same theory. A clue to an explanation arises from the following considerations. The helium nucleus is an extremely stable structure because there is no evidence that an α particle has ever been shattered. Consequently one would expect the change from 4H to He to involve a considerable evolution of energy and hence a disappearance of mass. A familiar theorem on electrical energy is "If any number of surfaces are fixed in position, and a given charge is placed on each surface, then the energy is a minimum when the charges are placed so that every surface is an equipotential"[3]. This suggests that if in Figure 1 the uniform surface distribution of a Lorentz electron be modified so as to become the surface distribution on a perfect spherical conductor of radius a, the negative mutual mass of the nucleus might be made sufficient to explain packing, and possibly more. This modification is clearly of importance only when the electron is very close to another charge, (such as in an atomic nucleus), so there is no reason for any anxiety in regard to the many beautiful results obtained in other lines with the Lorentz electron.

APPLICATION TO THE STATIC MODEL OF FIGURE 1.

Let us proceed to calculate the space average rest mass of two protons separated by an electron, treating them as equipotential spherical surface distributions.

First, we need an expression for the electric potential (in e.s.u.) at any point $P(r, \theta)$ produced by an electrified zone of surface density σ and angle α, $\alpha + d\alpha$, as shown in Figure 2. The potential at P produced by a spherical cap of angle α and uniform density σ is

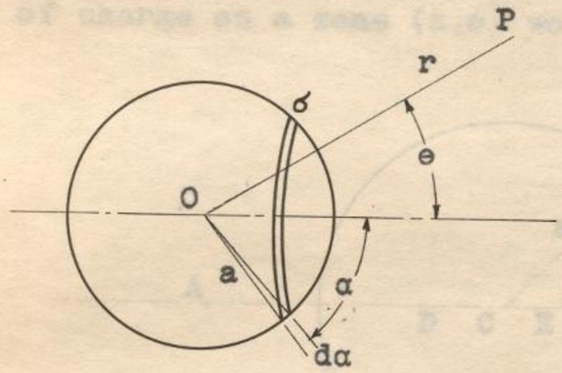


Figure 2

$$V_e = 2\pi a\sigma \left[(1-\cos\alpha)\frac{a}{r} + \sum_{1}^{\infty} \frac{P_{n-1}(\cos\alpha) - P_{n+1}(\cos\alpha)}{2n+1} \left(\frac{a}{r}\right)^{n+1} \times P_n(\cos\theta) \right], \quad r>a;$$

$$V_i = 2\pi a\sigma \left[(1-\cos\alpha) + \sum_{1}^{\infty} \frac{P_{n-1}(\cos\alpha) - P_{n+1}(\cos\alpha)}{2n+1} \left(\frac{r}{a}\right)^{n} \times P_n(\cos\theta) \right], \quad r<a.$$

$\frac{dV}{d\alpha}\, d\alpha$ is the potential dV at P due to a zone of angle α, $\alpha + d\alpha$ and surface density σ. This is

$$\frac{dV}{d\alpha}\, d\alpha = 2\pi a\sigma \sin\alpha \, d\alpha \sum_{0}^{\infty} \left(\frac{a}{r}\right)^{n+1} P_n(\cos\alpha)\, P_n(\cos\theta), \quad r>a;$$
$$= 2\pi a\sigma \sin\alpha \, d\alpha \sum_{0}^{\infty} \left(\frac{r}{a}\right)^{n} P_n(\cos\alpha)\, P_n(\cos\theta), \quad r<a. \qquad (2)$$

If the point P is on the sphere,

$$dV = \frac{dV}{d\alpha}\, d\alpha = 2\pi a\sigma \sin\alpha \, d\alpha \sum_{0}^{\infty} P_n(\cos\alpha)\, P_n(\cos\theta), \quad r=a. \qquad (2')$$

P_n stands for Legendre's n th coefficient.

~~$\left[\left(\frac{r}{a}\right)^0 = 1 \text{ when } r = 0\right]$.~~

Next we must find the surface density σ on the electron. Consider a conducting sphere with center at C, radius a, total charge -e. Point charges +e are at A and B. Images of A, B are at D, E of amount -ea/f each. (See Figure 3). If the sphere were earthed, the density of charge on a zone (a,θ) would be

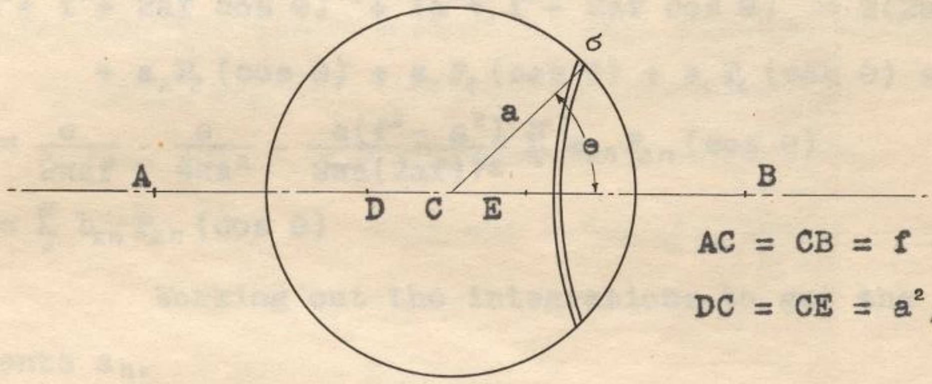


Figure 3

$$-\frac{e\ (f^2-a^2)}{4\pi a(a^2+f^2+2af\cos\theta)^{3/2}} - \frac{e\ (f^2-a^2)}{4\pi a(a^2+f^2-2af\cos\theta)^{3/2}} = u, \text{ say.}$$

$\int u\,dS = -2ea/f$. In order that the total charge on the electron be -e, a uniform charge $+2ea/f - e$, in addition to $\int u\,dS$, must be placed on the sphere. Therefore

$$\sigma = \frac{e}{2\pi af} - \frac{e}{4\pi a^2} - \frac{e\,(f^2-a^2)}{4\pi a}\left\{\frac{1}{(a^2+f^2+2af\cos\theta)^{3/2}} + \frac{1}{(a^2+f^2-2af\cos\theta)^{3/2}}\right\} \qquad (3)$$

$\int \sigma \, dS = -e$, as it should.

To expand σ in terms of Legendre's coefficients, put $\dfrac{1}{(a^2+f^2+2af\cos\theta)^{3/2}} = \dfrac{1}{(2af)^{3/2}(A+x)^{3/2}}$ where

$A = (a^2+f^2)/2af$ and $x = \cos\theta$.

$$(A+x)^{-3/2} = a_0 + a_1 P_1(x) + a_2 P_2(x) + a_3 P_3(x) + \ldots$$

By the usual process, $a_n = \frac{1}{2}(2n+1)\int_{-1}^{+1}(A+x)^{-3/2} P_n(x)\,dx$

$$(A-x)^{-3/2} = a_0 + a_1 P_1(-x) + a_2 P_2(-x) + a_3 P_3(-x) + \ldots$$
$$= a_0 - a_1 P_1(x) + a_2 P_2(x) - a_3 P_3(x) + \ldots$$

$$(a^2+f^2+2af\cos\theta)^{-3/2} + (a^2+f^2-2af\cos\theta) = 2(2af)^{-3/2}\left\{a_0 + a_2 P_2(\cos\theta) + a_4 P_4(\cos\theta) + a_6 P_6(\cos\theta) + \cdots\right\}$$

$$\sigma = \frac{e}{2\pi af} - \frac{e}{4\pi a^2} - \frac{e(f^2-a^2)}{2\pi a(2af)^{3/2}} \sum_0^\infty a_{2n} P_{2n}(\cos\theta)$$

$$= \sum_0^\infty b_{2n} P_{2n}(\cos\theta) \tag{4}$$

Working out the integrations to get the coefficients a_n,

$$\left.\begin{aligned} a_0 &= \sqrt{2af}\cdot 2a/(f^2-a^2) && \text{and} \\ a_n &= a_0(2n+1)(a/f)^n, && n = 0, 2, 4, 6, \ldots \text{ or} \\ a_{2n} &= a_0(4n+1)(a/f)^n, && n = 0, 1, 2, 3, \ldots \end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned} b_0 &= e/2\pi af - e/4\pi a^2 - e(f^2-a^2)a_0/2\pi a(2af)^{3/2} = -e/4\pi a^2 \\ b_{2n} &= b_0 2(4n+1)(a/f)^{2n+1}, && n = 0, 1, 2, 3, \end{aligned}\right\} \tag{6}$$

For convenience, write $a:f = Y$

$$\sum_0^\infty b_{2n}^2/(4n+1)^2 = b_0^2 + \sum_1^\infty b_{2n}^2/(4n+1)^2 = b_0^2 + 4b_0^2(Y^6 + Y^{10} + Y^{14} + Y^{18} + \ldots)$$
$$= b_0^2\left[1 + 4Y^6/(1-Y^4)\right] \tag{7}$$

an expression which is needed later on.

We are now ready to compute the double surface integral $\iint \frac{de_1 de_2}{r}$ over the surface of the electron. de_1 and de_2 are elements of charge on the electron, r the distance between them. Suppose de_1 lies on a zone of angle θ_1, $\theta_1 + d\theta_1$, and de_2 on a zone of angle θ_2, $\theta_2 + d\theta_2$. Let σ_1 and σ_2 be the surface density for the angles θ_1 and θ_2 respectively. Then put

$de_1 = \sigma_1 a^2 \sin\theta_1 \, d\theta_1 \, d\phi_1$ and $de_2 = \sigma_2 a^2 \sin\theta_2 \, d\theta_2 \, d\phi_2$

Now de_2/r integrated with respect to ϕ_2 is merely the expression for the potential at de_1 produced by the zone θ_2, $\theta_2 + d\theta_2$, and this is given by equation (2). Put $\mu_1 = \cos\theta_1$ and $\mu_2 = \cos\theta_2$ wherever convenient.

$$\iint \frac{de_1 de_2}{r} = \iint \int_{\theta_2=0}^{\pi} de_1 \, 2\pi a \sigma_2 \sin\theta_2 \, d\theta_2 \sum_0^\infty P_n(\cos\theta_1) P_n(\cos\theta_2)$$

$$= 4\pi^2 a^3 \int_{-1}^{1}\int_{-1}^{1} d\mu_1 \, d\mu_2 \cdot \sum_0^\infty b_{2n} P_{2n}(\mu_1) \cdot \sum_0^\infty b_{2n} P_{2n}(\mu_2) \cdot \sum_0^\infty P_n(\mu_1) P_n(\mu_2) \qquad 5$$

$$= 4\pi^2 a^3 \int_{-1}^{1} d\mu_2 \sum_0^\infty b_{2n} P_{2n}(\mu_2) \cdot \sum_0^\infty \frac{2b_{2n}}{4n+1} P_{2n}(u_2)$$

$$= 4\pi^2 a^3 \sum_0^\infty b_{2n}^2 \left(\frac{2}{4n+1}\right)^2 = 16\pi^2 a^3 \sum_0^\infty \frac{b_{2n}^2}{(4n+1)^2}$$

$$= 16\pi^2 a^3 \frac{e^2}{16\pi^2 a^4}\left[1 + 4\frac{Y^6}{1-Y^4}\right]$$

$$= \frac{e^2}{a}\left[1 + 4\frac{Y^6}{1-Y^4}\right] \qquad (8)$$

With these preliminaries we are now ready to find the space average rest mass of the electron. This is known to be

$$\frac{1}{6\pi c^2} \iint \frac{de_1 de_2}{r} \qquad 4 \qquad (9)$$

for any sort of a charge. For our particular electron, we have calculated this integral. Inserting its value as given in (8), we get

$$\frac{e^2}{6\pi a c^2}\left[1 + \frac{4Y^6}{1-Y^4}\right] \qquad (10)$$

for the intrinsic space average rest mass of our electron. Of course this reduces to $e^2/6\pi ac^2$ when the distance f becomes very great, for then $Y = a{:}f$ becomes zero.

Seeing that the mass of this electron has been increased by postulating that it is to be an equipotential surface, rather than a uniform distribution, it is natural to enquire into possible changes, caused by the same assumption, in the mass of the **two** protons A and B. The easiest way to dispose of this question is to treat the protons as if they were in a uniform field F. Proton B is acted upon by charges e at A, $-ea/f$ at D and E, $2ea/f-e$ at C. $AB = 2f$, $DB = f + a^2/f$. $CB = f$, $EB = f - a^2/f$.

$$\text{So } F = \frac{e}{4\pi}\left\{\frac{1}{4f^2} - \frac{a}{f(f + a^2/f)^2} - \frac{a}{f(f - a^2/f)^2} + \frac{2a}{f^3} - \frac{1}{f^2}\right\} \quad (11)$$

F can never be infinite because the center of a proton can not touch the surface of an electron. With the aid of Figure 4, consider proton B as a conducting sphere of radius A and charge e in a uniform field F. The potential at P(r,e)is

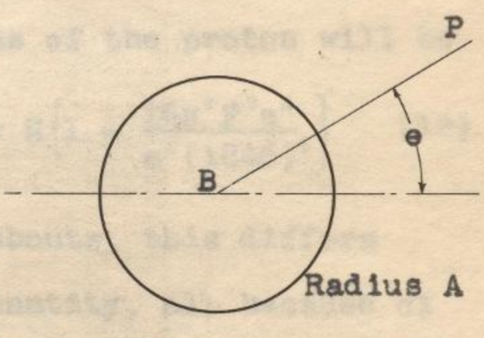


Figure 4

$$V = -F(r - A^3/r^2)\cos\theta \quad + \frac{e}{4\pi r}$$

The radial and tangential components of the resulting field at P are

$$-\frac{dV}{r\,d\theta} = \frac{-F}{r}(r - A^3/r^2)\sin\theta$$

$$-\frac{dV}{dr} = -F(1 + 2A^3/r^3)\cos\theta + e/4\pi r^2$$

On the surface where $r = A$ they are 0 and $-3F \cos\theta + e/4\pi A^2$ (and 0) respectively. Consequently the density of electrification at any point (A, θ) on the proton is

$$w = -3F\cos\theta + \frac{e}{4\pi A^2} = H\cos\theta + K = H\,P_1(\cos\theta) + K\,P_0(\cos\theta)$$

Just as in dealing with the electron, take two elements of charge $de_1 = w_1 A^2 \sin\theta_1\, d\theta_1\, d\phi_1$ and $de_2 = w_2 A^2 \sin\theta_2\, d\theta_2\, d\phi_2$ at points where θ is θ_1 and θ_2 respectively. Again write $\mu_1 = \cos\theta_1$ and $\mu_2 = \cos\theta_2$. Then over this surface

$$\iint \frac{de_1 de_2}{r} = \iint \int_{-1}^{+1} de_1\; 2\pi A w_2\, d\mu_2 \sum_0^\infty P_n(\mu_1)\, P_n(\mu_2)$$

$$= 4\pi^2 A^3 \int_{-1}^{1}\int_{-1}^{1} d\mu_1 d\mu_2 \left\{K\,P_0(\mu_1) + H\,P_1(\mu_1)\right\}\left\{K\,P_0(\mu_2) + H\,P_1(\mu_2)\right\} \times \sum_0^\infty P_n(\mu_1)\, P_n(\mu_2)$$

$$= 4\pi^2 A^3 \int_{-1}^{1} d\mu_2 \left\{K\,P_0(\mu_2) + H\,P_1(\mu_2)\right\}\left\{2K\,P_0(\mu_2) + \tfrac{2}{3} H\,P_1(\mu_2)\right\}$$

$$= 4\pi^2 A^3 \left\{4K^2 + 4H^2/9\right\} = 4\pi^2 A^3 (4e^2/16\pi^2 A^4 + 36F^2/9)$$

$$= e^2/A + 16\pi^2 F^2 A^3$$

So the space average rest mass of the proton will be

$$\frac{1}{6\pi c^2}\iint \frac{de_1 de_2}{r} = \frac{e^2}{6\pi A c^2}\left(1 + \frac{16\pi^2 F^2 A^4}{e^2}\right) = M\left\{1 + \frac{16\pi^2 F^2 a^4}{e^2 (1846)^4}\right\} \qquad (12)$$

Unless Y is as high as .998 or thereabouts, this differs from M only by an insensibly small quantity, all because of the small radius $A = a/1846$. Complications arise when Y is extremely close to unity, so these high values of Y will be left out entirely from the discussion, and the protons will be treated as having mass M.

It should also be noted that on account of the comparatively small radius of the protons, secondary images were neglected, that is, the protons can be treated as if they are uniformly charged spheres, hence as point charges in finding their effect at points external to them.

The space average intrinsic rest mass of the half nucleus is therefore

$$2M + m(1+4\frac{Y^6}{1-Y^4}) = 2 \times 1846\ m + m(1+4\frac{Y^6}{1-Y^4}) \qquad (13)$$

To this must be added the mutual mass of the half nucleus which is given by expression (1). This can be resolved into six partial summations, for convenience.

$$\text{Mutual Mass} = \frac{1}{6\pi c^2}\sum_{ij}\frac{e_i e_j}{r_{ij}}, \qquad i \neq j \qquad (1)$$

(i) Let e_i be proton A and e_j be the charge of the zone θ, $\theta + d\theta$ on the electron. $e_i = e$, $e_j = \sigma\, dS = 2\pi a\, \sigma \sin\theta\, d\theta$, and $r_{ij} = \sqrt{(a^2 + f^2 + 2af\cos\theta)}$. This partial summation is $e\iint(a^2 + f^2 + 2af\cos\theta)^{-1/2}\,\sigma\, dS$

$$= \left(\frac{e^2}{2\pi af} - \frac{e^2}{4\pi a^2}\right)\int_0^\pi \frac{2\pi a^2 \sin\theta\, d\theta}{(a^2 + f^2 + 2af\cos\theta)^{1/2}}$$

$$- \frac{e^2(f^2 - a^2)}{4\pi a}\int_0^\pi \frac{2\pi a^2 \sin\theta\, d\theta}{(a^2 + f^2 + 2af\cos\theta)^2}$$

$$- \frac{e^2(f^2 - a^2)}{4\pi a}\int_0^\pi \frac{2\pi a^2 \sin\theta\, d\theta}{(a^2 + f^2 + 2af\cos\theta)^{1/2}(a^2 + f^2 - 2af\cos\theta)^{3/2}}$$

$$= -e^2/f + 2ae^2/f - e^2 a/(f^2 - a^2) - e^2 a/(f^2 + a^2) \qquad (14)$$

As might have been anticipated, this is the same result as would have been obtained on replacing the electron by charges $-ea/f$ at D and E, and $2ea/f - e$ at C.

(ii) Let e_i be proton A and e_j be proton B. This partial summation is simply $e^2/2f$.

(iii) Let e_i be the charge on the zone θ, $\theta + d\theta$ and e_j be proton B. This gives the same result as (i).

A

(iv) Let e_i be the charge on the zone θ, $\theta + d\theta$ and e_j be the proton B. This gives the same result as (i).

(v) Let e_i be proton B and e_j be proton A. This gives the same result as (ii).

(vi) Let e_i be proton B and e_j be the charge on the zone θ, $\theta + d\theta$. This gives the same result as (i).

By adding these up, the result is for the mutual mass,

$$\frac{1}{6\pi c^2}\sum_{ij}\frac{e_i e_j}{r_{ij}} = \frac{1}{6\pi c^2}\left\{4\left(-e^2/f + 2ae^2/f - e^2a/(f^2+a^2) - e^2a/(f^2-a^2)\right) + e^2/f\right\}$$

$$= \frac{-e^2}{6\pi ac^2}\left\{3a/f - 8a^2/f^2 + 8a^2f^2/(f^2-a^2)\right\}$$

$$= -m\left\{3Y - 8Y^2 + 8Y^2/(1-Y^4)\right\} \qquad (15)$$

The mass of the entire helium atom is

$$\text{Intrinsic Mass} + \text{Mutual Mass} = 2\left\{m + 2 \times 1846\ m + m\left(1+4\frac{Y^6}{1-Y^4}\right) - m\left(3Y - 8Y^2 + 8\frac{Y^2}{1-Y^4}\right)\right\}$$

which must amount to $\frac{4}{4 \times 1.0078} \times 4m \times 1847 =$

$$= 4m \times 1847(1 - .00774\ldots)$$

i.e. $$F(Y) \equiv 2\left(4\frac{Y^6}{1-Y^4} - 3Y + 8Y^2 - 8\frac{Y^2}{1-Y^4}\right) = -\left(\frac{8Y^6}{1-Y^4} + 6Y\right)$$

$$= -4 \times 1847 \times .00774\ldots = -57.18$$

$$\doteq -57 \text{ closely enough.} \qquad (16)$$

When cleared of fractions this becomes an equation of the sixth degree,

$$8Y^6 - 6Y^5 + 57Y^4 + 6Y - 57 = 0 \qquad (17)$$

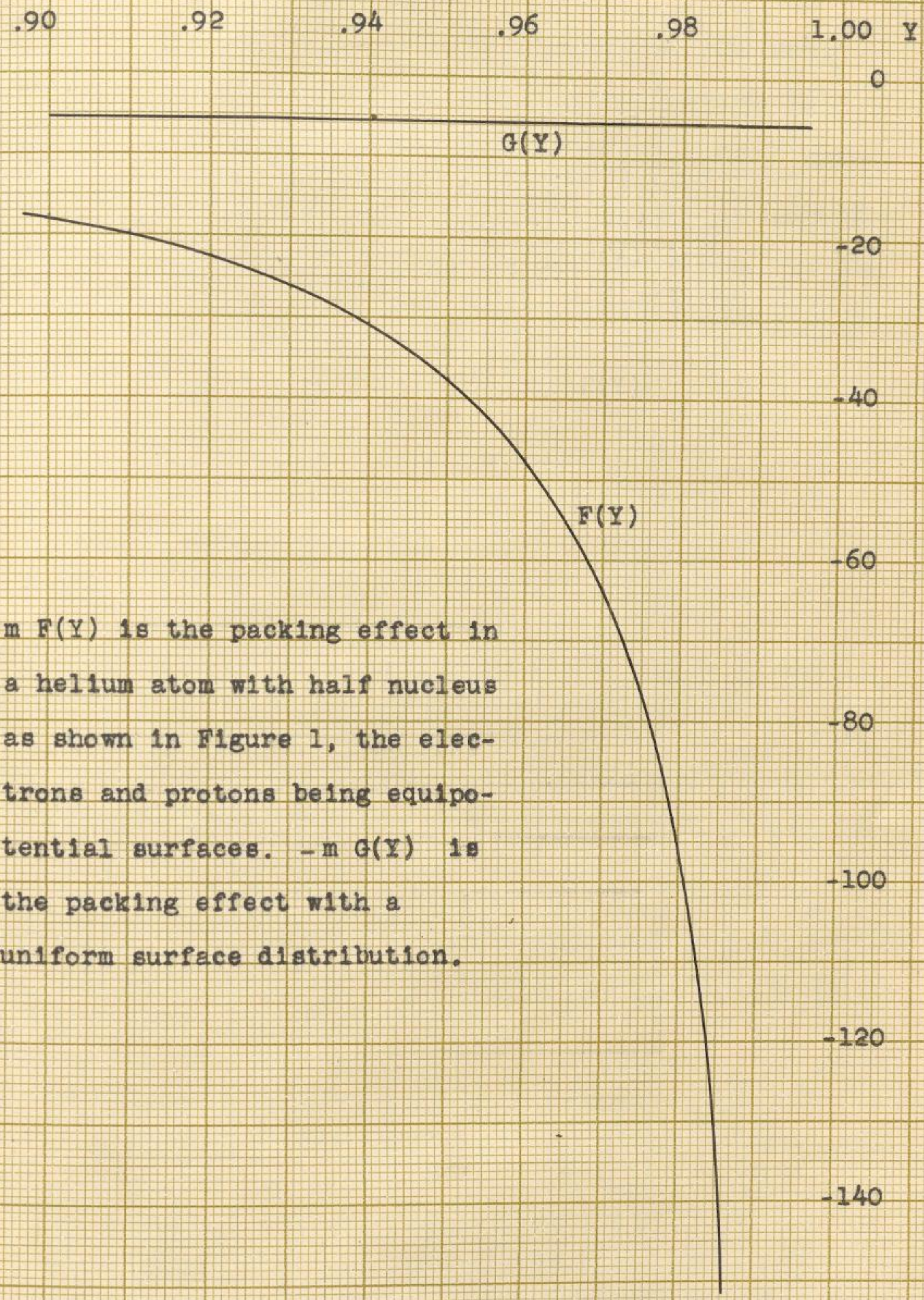
This equation has one positive root. It is $Y = .966249\ldots$ $Y^{-1} = 1.0349\ldots$ So if $f = 1.035\ a$ the requisite amount of packing is obtained. The separation between the center of either proton and the surface of the electron is 0.035 a, which is 65 times the radius of a proton.

m F(Y) is the resultant negative mass of the helium atom, that is, 4 x 1847m + m F(Y) is the mass of the helium atom. It is interesting to plot F(Y) along with the corresponding packing effect that an ordinary Lorentz electron would give — call this m G(Y).

$$G(Y) = -6Y \tag{18}$$

These are plotted in Figure 5. This graph shows that as the distance f decreases the packing m F(Y) increases almost without limit.

Thus it appears that the idea of treating electrons and protons as equipotential spheres rather than uniformly charged ones works out very well. This modification can be expected to increase considerably the packing effect over that given by ordinary Lorentz electrons, regardless of the model adopted for the nucleus, if the statements made in the opening paragraphs be considered a general proof. In order to test the idea, <u>some</u> model must be considered. That of Figure 1 is probably unequalled for simplicity and definiteness, because the expression found for σ is exact and not merely a first approximation, as there are no multiple images when the protons are treated as points and no other electron is near. Unfortu-



-m F(Y) is the packing effect in a helium atom with half nucleus as shown in Figure 1, the electrons and protons being equipotential surfaces. -m G(Y) is the packing effect with a uniform surface distribution.

Figure 5

nately this is not considered the best model, the main reason being that the helium nucleus or α particle is probably a unit. The fact that the assumption here introduced apparently yields such good results is some justification for it; but as long as the true arrangement of the electrons and protons in the nucleus remains unknown there is little chance to give it a real test.

APPLICATION TO THE STATIC LENZ MODEL

The inverted oxygen model proposed by W. Lenz[6] will now be used to test the idea of equipotential electrons. It is shown in Figure 6. A, B, C, D are four protons, E, F the centers of the two electrons. A', B', C', D' and A", B", C", D" are the images of the protons in the electrons. F' is the image in E of the uniform distribution on F, and F" is the image in F of that on E. No account will be taken of secondary images.

Let $EA = EB = EC = ED = EA'' = \cdots = f = a/Y$

$$Y = a:f, \qquad Z = a:EF, \qquad \Delta = \measuredangle AEF = \measuredangle BEF = \cdots$$

$$AE = a/Y, \qquad EF = a/Z, \qquad AO = \frac{a}{2Z} \tan\Delta$$

$$AB = \frac{a}{\sqrt{2}\,Z} \tan\Delta, \qquad AC = aZ^{-1}\tan\Delta, \qquad A'E = aY$$

$$AA' = aY^{-1}(1-Y^2) \qquad AB' = aY^{-1}(1+Y^4-2Y^2\cos^2\Delta)^{1/2}$$

$$EF' = aZ, \qquad EF'' = aZ^{-1}(1-Z^2)$$

$$EA'' = aZ^{-1}(1+Y^2Z^2-Y^2)^{1/2} \qquad AC' = aY^{-1}(1+Y^4-2Y^2\cos 2\Delta)^{\frac{1}{2}}$$

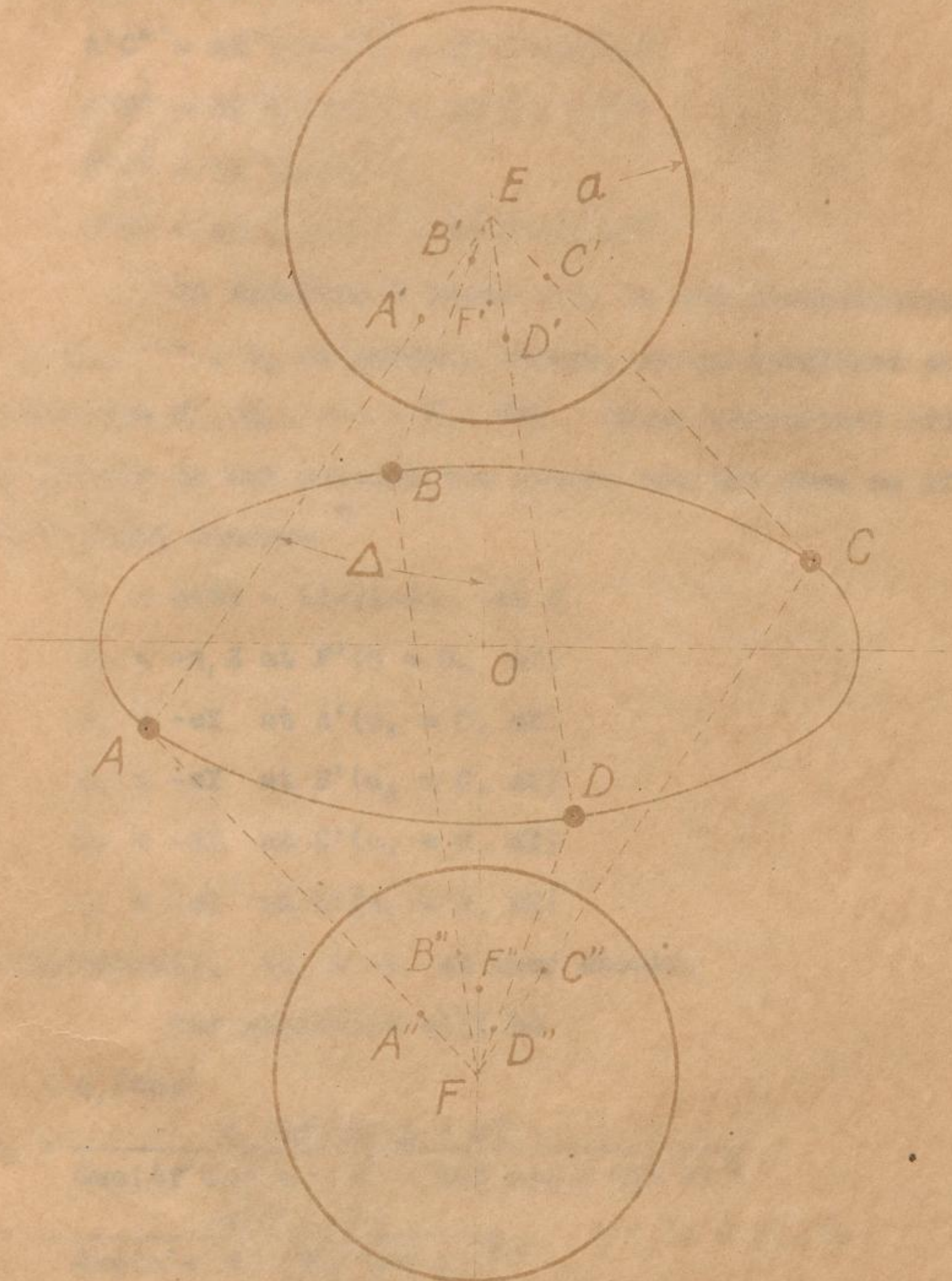


FIGURE 6

$$A'A'' = aZ^{-1}(1-Y^2)$$
$$A'C'' = aZ^{-1}([1-Y^2]^2 + 4Y^2Z^2\sin^2\Delta)^{1/2}$$
$$A'F'' = aZ^{-1}([1-Z^2]^2 + 2Y^2Z^2 - Y^2)^{1/2}$$
$$F'F'' = aZ^{-1}(1-2Z^2),$$
$$A'B'' = aZ^{-1}([1-Y^2]^2 + 2Y^2Z^2\sin^2\Delta)^{1/2}$$

On electron E there will be six distributions S_1, S_2, $\cdots$, S_6 of surface charge, whose densities at any point are σ_1, σ_2, $\cdots$, σ_6, say. Their electrical effects at points on and outside the sphere are the same as if they were point charges[7]

$$
\begin{aligned}
q_1 &= e(4Y - 1)/(1-Z) \quad \text{at } E \\
q_2 &= -q_1 Z \text{ at } F'(\theta = 0,\ aZ) \\
q_3 &= -eY \quad \text{at } A'(\alpha_3 = 0,\ aY) \\
q_4 &= -eY \quad \text{at } B'(\alpha_4 = 0,\ aY) \\
q_5 &= -eY \quad \text{at } C'(\alpha_5 = 0,\ aY) \\
q_6 &= -eY \quad \text{at } D'(\alpha_6 = 0,\ aY)
\end{aligned}
\qquad (19)
$$

respectively. $\Sigma q_i = -e$ as they should.

The densities will be

$$\sigma_1 = q_1/4\pi a^2$$

$$\sigma_2 = \frac{q_1(4f^2\cos^2\Delta - a^2)}{4\pi a(4f^2\cos^2\Delta + a^2 - 4af\cos\Delta\cos\theta)^{3/2}}$$

$$\sigma_i = \frac{-e(f^2 - a^2)}{4\pi a(a^2 + f^2 - 2af\cos\alpha_i)^{3/2}}, \quad i = 3, 4, 5, 6.$$

the polar axes $\theta = 0$ being the line EF, and $\alpha_i = 0$ being EA, EB, EC, ED for $i = 3, 4, 5, 6$.

$$\left(\frac{f^2 + a^2}{2af} - \cos\alpha_i\right)^{-3/2} = (A - x)^{-3/2} = \sum_0^\infty a_n P_n(x)$$

where $A = (f^2 + a^2)/2af$ and $x = \cos\alpha_i$

$$a_n = \tfrac{1}{2}(2n+1)\int_{-1}^{1}(A-x)^{-3/2}P_n(x)dx = a_0(2n+1)Y^n, \quad n \neq 0$$

$$a_0 = \frac{\sqrt{2af}}{f^2 - a^2}\, 2a$$

$$\sigma_i = \frac{-e(f^2 - a^2)}{4\pi a(a^2 + f^2 - 2af \cos \alpha_i)^{3/2}} = \frac{-e(f^2 - a^2)}{4\pi a(2af)^{3/2}} \cdot \frac{1}{\left(\frac{f^2 + a^2}{2af} - \cos \alpha_i\right)^{3/2}}$$

$$= -\frac{e}{4\pi a^2} Y\left\{1 + 3Y\, P_1(\cos \alpha_i) + 5Y^2 P_2(\cos \alpha_i) + \cdots \right\}$$

$$= -\frac{e}{4\pi a^2} Y \sum_0^\infty (2n+1)\, Y^n P_n(\cos \alpha_i)\,, \quad i = 3, 4, 5, 6, \qquad (20)$$

To get σ_2, replace α_i by θ, Y by Z, $-e$ by $-q_2$.

$$\sigma_2 = \frac{-q_1}{4\pi a^2} Z \sum_0^\infty (2n+1)\, Z^n P_n(\cos \theta) \qquad (21)$$

To calculate $\iint \frac{de_1\, de_2}{r}$ over electron E.

This integration may be conveniently divided into six parts, corresponding to de_1 as an element of S_1, S_2, ..., or S_6. Let de be an element of S_i and r_j the distance from de to q_j.

$$\iint \frac{de_1\, de_2}{r} = \sum_{i=1}^{6} \iint \left(\frac{de}{r_1} q_1 + \frac{de}{r_2} q_2 + \; \cdots\cdots + \frac{de}{r_6} q_6\right) \qquad (22)$$

Use a, θ, ϕ for the coordinates of de.

$$de = a^2 \sigma_i \sin\theta\, d\theta\, d\phi, \qquad \sigma_i = \sigma_i(\theta).$$

Again use $\cos\theta \equiv \mu$ when convenient.

$\iint_{S_i} \frac{de}{r_j}$ is the mathematical expression for the potential at q_j produced by the surface distribution S_i. If h_j, β_j are the coordinates of q_j (the axes $\beta_j = 0$ and $\theta = 0$ being coincident), equation (2) gives

$$q_j \iint_{S_i} \frac{de}{r_j} = 2\pi a \int_{-1}^{+1} \sigma_i\, d\mu \sum_{j=1}^{6} \sum_0^\infty q_j \left(\frac{h_j}{a}\right)^n P_n(\cos \beta_j) P_n(\mu) = I_i \text{ say.}$$

$$\iint \frac{de_1\, de_2}{r} = \sum_{i=1}^{6} I_i$$

$$I_1 = 2\pi a \int_{-1}^{+1} \frac{q_1}{4\pi a^2}\, d\mu \left\{ q_1 + q_2 \sum_0^\infty Z^n P_n(\mu) \cdot P_n(1) \right.$$

$$\left. + 4q_3 \sum_0^\infty Y^n P_n(\mu)\, P_n(\cos \Delta) \right\} = \frac{q_1}{a}(q_1 + q_2 + 4q_3)$$

$$I_2 = 2\pi a \int_{-1}^{+1} d\mu \, \frac{-q_1 Z}{4\pi a^2} \sum_0^\infty (2n+1) \, Z^n P_n(\mu) \Big\{ q_1 + q_2 \sum_0^\infty Z^n P_n(\mu) \, P_n(1)$$

$$+ 4q_3 \sum_0^\infty Y^n P_n(\mu) \, P_n(\cos\Delta) \Big\}$$

$$= -\frac{q_1 Z}{a} \Big\{ q_1 + q_2 \sum_0^\infty Z^{2n} P_n(1) + 4q_3 \sum_0^\infty Y^n Z^n P_n(\cos\Delta) \Big\}$$

$$= -\frac{q_1 Z}{a} \Big\{ q_1 + q_2/(1-Z^2) + 4q_3/\sqrt{1+Y^2 Z^2 - 2YZ\cos\Delta} \Big\}$$

$$= -\frac{q_1 Z}{a} \Big\{ q_1 + q_2/(1-Z^2) + 4q_3/\sqrt{1+Y^2 Z^2 - Y^2} \Big\},$$

remembering that $(1 + r^2 - 2rx)^{-1/2} = \sum_0^\infty r^n P_n(x)$.

$$I_3 = 2\pi a \int_{-1}^{+1} d(\cos\alpha_3) \cdot \frac{-e}{4\pi a^2} \cdot Y \sum_0^\infty (2n+1) \, Y^n P_n(\cos\alpha_3) \quad \times$$

$$\Big\{ q_1 + q_2 \sum_0^\infty Z^n P_n(\cos\alpha_3) \, P_n(\cos\Delta) + q_3 \sum_0^\infty Y^n P_n(\cos\alpha_3) \, P_n(1)$$

$$+ q_4 \sum_0^\infty Y^n P_n(\cos\alpha_3) \, P_n(\cos^2\Delta) + q_5 \sum_0^\infty Y^n P_n(\cos\alpha_3) \, P_n(\cos 2\Delta)$$

$$+ q_6 \sum_0^\infty Y^n P_n(\cos\alpha_3) \, P_n(\cos^2\Delta) \Big\}$$

$$= \frac{-eY}{a} \Big\{ q_1 + q_2 \sum_0^\infty Y^n Z^n P_n(\cos\Delta) \, P_n(1) + q_3 \sum_0^\infty Y^{2n} P_n^2(1)$$

$$+ 2q_4 \sum_0^\infty Y^{2n} P_n(1) \, P_n(\cos^2\Delta) + q_5 \sum_0^\infty Y^{2n} P_n(1) \, P_n(\cos 2\Delta) \Big\}$$

$$= -\frac{eY}{a} \Big\{ q_1 + q_2/\sqrt{1+Y^2 Z^2 - Y^2} + q_3/(1-Y^2)$$

$$+ 2q_3/\sqrt{1+Y^4 - 2Y^2\cos^2\Delta} + q_3/\sqrt{1+Y^4-2Y^2\cos 2\Delta} \Big\}$$

$$= I_4 = I_5 = I_6 .$$

The space average intrinsic rest mass of each electron is therefore

$$\frac{1}{6\pi c^2} \iint \frac{de_1 de_2}{r} = \frac{1}{6\pi c^2} \sum_1^6 I_i = \frac{1}{6\pi a c^2} \Big\{ q_1^2 (1-Z) + q_1 q_2$$

$$+ 8q_1 q_3 - q_1 q_2 Z/(1-Z^2) - (4q_1 q_3 Z + 4q_2 eY)/\sqrt{1+Y^2 Z^2 - Y^2}$$

$$-4q_3 eY/(1-Y^2) - 8eq_3 Y/\sqrt{1+Y^4-2Y^2\cos^2\Delta}$$

$$- 4eq_3 Y/\sqrt{1+Y^4-2Y^2\cos 2\Delta} \Big\}$$

$$= m + m \Big\{ -1 + (4Y-1)^2/(1-Z) - Z(4Y-1)^2/(1-Z)^2$$

$$- 8Y(4Y-1)/(1-Z) + Z^2(4Y-1)^2/(1-Z)^2(1-Z^2)$$

$$+ 8YZ(4Y-1)/(1-Z)\sqrt{1+Y^2Z^2-Y^2} + 4Y^2/(1-Y^2)$$

$$\left. + 8Y^2/\sqrt{1+Y^4-2Y^2\cos^2\Delta} + 4Y^2/\sqrt{1+Y^4-2Y^2\cos 2\Delta} \right\}$$

$$= m + m\,\tfrac{1}{2}I(Y,Z) \qquad (23)$$

where I(Y,Z) denotes twice the quantity enclosed by the braces. I(Y,Z) vanishes when Y and Z are both zero.

To find the mutual mass of this nucleus.

For convenience, let (A,B) denote $q_A q_B/AB$, that is, the charge effective at A multiplied by the charge effective at B divided by the distance AB. (A,B) = (B,A). The space average mutual mass of the nucleus will be

$$\frac{1}{6\pi c^2}\,\Sigma\,\Sigma(A,\ B) \qquad A \neq B$$

and in which A and B never lie inside the same sphere. This will be a sum of 180 terms, not all different.

1. $(E,\ A) = q_1 e/aY^{-1} = (E,\ B) = (E,\ C) = E,\ D)$
2. $(E,\ F) = q_1^2/aZ^{-1}$
3. $(E,\ F'') = q_1 q_2/aZ^{-1}(1-Z^2)$
4. $(E,\ A'') = q_1 q_3/aZ^{-1}\sqrt{1-Y^2+Y^2Z^2} = (E,B'') = (E,C'') = (E,D'')$
5. $(F',F'') = q_2^2/aZ^{-1}(1-2Z^2)$
6. $(F',A) = q_2 e/aY^{-1}\sqrt{1+Y^2Z^2-Y^2} = (F',B) = (F',C) = (F',D)$
7. $(F',F) = (E,F'')$ already found.
8. $(F',A'') = q_2 q_3/aZ^{-1}\sqrt{(1-Z^2)^2+2Y^2Z^2-Y^2} = (F',B'') = (F',C'')$ $= (F',D'')$
9. $(A',A) = q_3 e/aY^{-1}(1-Y^2)$
10. $(A',B) = q_3 e/aY^{-1}\sqrt{1+Y^4-2Y^2\cos^2\Delta} = (A',D)$
11. $(A',C) = q_3 e/aY^{-1}\sqrt{1+Y^4-2Y^2\cos 2\Delta}$

12. $(A',F) = (E,A'')$ already found.

13. $(A',F'') = (F',A'')$ already found.

14. $(A',A'') = q_3^2/aZ^{-1}(1-Y^2)$

15. $(A',B'') = q_3^2/aZ^{-1}\sqrt{(1-Y^2)^2 + 2Y^2Z^2\sin^2\Delta} = (A',D'')$

16. $(A',C'') = q_3^2/aZ^{-1}\sqrt{(1-Y^2)^2 + 4Y^2Z^2\sin^2\Delta}$

Terms 9 to 16 hold with B' in place of A'.

Terms 9 to 16 hold with C' in place of A'.

Terms 9 to 16 hold with D' in place of A'.

17. $(A,E) = (E,A)$ already found.

18. $(A,F') = (F',A)$ already found.

19. $(A,A') = (A',A)$ already found.

20. $(A,B') = (A',A)$ already found, $= (A,D')$

21. $(A,C') = (A',C)$ already found.

22. $(A,B) = e^2/\frac{a}{\sqrt{2}\,Z} \tan\Delta = (A,D)$

23. $(A,C) = e^2/\frac{a}{Z} \tan\Delta$

24. $(A,F) = (F',A)$ already found.

25. $(A,F'') = (F',A)$ already found.

26. $(A,A'') = (A',A)$ already found.

27. $(A,B'') = (A',B)$ already found, $= (A,D'')$

28. $(A,C'') = (A',C)$ already found.

Terms 17 to 28 inclusive hold with B in place of A.

Terms 17 to 28 inclusive hold with C in place of A.

Terms 17 to 28 inclusive hold with D in place of A.

Terms 1 to 4 inclusive hold with F in place of E.

Terms 5 to 8 inclusive hold with F" in place of F'.

Terms 9 to 16 inclusive hold with A" in place of A'.

Terms 9 to 16 inclusive hold with B" in place of A'.

Terms 9 to 16 inclusive hold with D" in place of A'.

$$\text{Mutual Mass} = \frac{1}{6\pi c^2}\Big[2\Big\{4(E,A) + (E,F) + (E,F'') + 4(E,A'')\Big\} + 2\Big\{(F',F'') + 4(F',A) + (F',F) + 4(F',A'')\Big\} + 8\Big\{(A',A) + 2(A',B) + (A',C) + (A',F) + (A',F'') + (A',A'') + 2(A',B'') + (A',C'')\Big\} + 4\Big\{2(E,A) + 2(F',A) + 2(A',A) + 4(A',B) + 2(A',C) + 2(A,B) + (A,C)\Big\}\Big]$$

$$= \frac{2}{6\pi c^2}\Big[8(E,A) + (E,F) + 2(E,F'') + 8(E,A'') + (F',F'') + 8(F',A) + 8(F',A'') + 8(A',A) + 16(A',B) + 8(A',C) + 4(A',A'') + 8(A',B'') + 4(A',C'') + 4(A,B) + 2(A,C)\Big]$$

$$= \frac{2}{6\pi a c^2}\Big\{8q_1 eY + q_1^2 Z + 2q_1 q_2 Z/(1-Z^2) + 8q_1 q_3 Z/\sqrt{1+Y^2Z^2-Y^2} + q_2^2 Z/(1-2Z^2) + 8q_2 eY/\sqrt{1+Y^2Z^2-Y^2} + 8q_2 q_3 Z/\sqrt{(1-Z^2)^2+2Y^2Z^2-Y^2} + 8eq_3 Y/(1-Y^2) + 16\, eq_3 Y/\sqrt{1+Y^4-2Y^2\cos^2\Delta} + 8\, eq_3 Y/\sqrt{1+Y^4-2Y^2\cos 2\Delta} + 4q_3^2 Z/(1-Y^2) + 8q_3^2 Z/\sqrt{(1-Y^2)^2+2Y^2Z^2\sin^2\Delta} + 4q_3^2 Z/\sqrt{(1-Y^2)^2+4Y^2Z^2\sin^2\Delta} + 4e^2 Z\sqrt{2}\cot\Delta + 2e^2 Z\cot\Delta\Big\}$$

$$= 2m\Big\{\frac{8Y(4Y-1)}{1-Z} + \frac{(4Y-1)^2 Z}{(1-Z)^2} - \frac{2Z^2(4Y-1)^2}{(1-Z^2)(1-Z)^2} - \frac{8YZ(4Y-1)}{(1-Z)\sqrt{1+Y^2Z^2-Y^2}} + \frac{Z^3(4Y-1)^2}{(1-Z)^2(1-2Z^2)} - \frac{8YZ(4Y-1)}{(1-Z)\sqrt{1+Y^2Z^2-Y^2}} + \frac{8YZ(4Y-1)}{(1-Z)\sqrt{(1-Z^2)^2+2Y^2Z^2-Y^2}} - \frac{8Y^2}{1-Y^2} - \frac{16Y^2}{\sqrt{1+Y^4-2Y^2\cos^2\Delta}} - \frac{8Y^2}{\sqrt{1+Y^4-2Y^2\cos 2\Delta}} + \frac{4Y^2Z}{1-Y^2}$$

$$+ \frac{8Y^2Z}{\sqrt{(1-Y^2)^2 + 2Y^2Z^2\sin^2\Delta}} + \frac{4Y^2Z}{\sqrt{(1-Y^2)^2 + 4Y^2Z^2\sin^2\Delta}}$$

$$\left. + 4Z\sqrt{2}\cot\Delta + 2Z\cot\Delta \right\} \equiv m\,M(Y,Z) \qquad (24)$$

$M(Y,Z)$ vanishes when Y and Z become zero.

The intrinsic mass of the entire atom is

$4 \times 1846 + 2m + m\left\{2 + I(Y,Z)\right\}$

The intrinsic mass plus mutual mass is

$4(1846 + 1)\,m + m\left[I(Y,Z) + M(Y,Z)\right]$

which must amount to $4 \times 1847\,m(1-.00774) = 4 \times 1847m - 57m$.

So $I(Y,Z) + M(Y,Z) = -57$ to account for the packing.

Collecting terms,

$$I(Y,Z) + M(Y,Z) = 2\left[-1 + \frac{(4Y-1)^2}{1-Z} - \frac{Z^2(4Y-1)^2}{(1-Z^2)(1-Z)^2}\right.$$

$$- \frac{4Y^2(1-Z)}{1-Y^2} - \frac{8Y^2}{\sqrt{1+Y^4-2Y^2\cos^2\Delta}} - \frac{4Y^2}{\sqrt{1+Y^4-2Y^2\cos 2\Delta}}$$

$$- \frac{8YZ(4Y-1)}{(1-Z)\sqrt{1+Y^2Z^2-Y^2}} + \frac{Z^3(4Y-1)^2}{(1-Z)^2(1-2Z^2)}$$

$$+ \frac{8YZ^2(4Y-1)}{(1-Z)\sqrt{(1-Z^2)^2 + 2Y^2Z^2-Y^2}} + \frac{8Y^2Z}{\sqrt{(1-Y^2)^2 + 2Y^2Z^2\sin^2\Delta}}$$

$$\left. + \frac{4Y^2Z}{\sqrt{(1-Y^2)^2 + 4Y^2Z^2\sin^2\Delta}} + 2Z(2\sqrt{2} + 1)\cot\Delta\right] \qquad (25)$$

The greatest packing effect is obtained when the electrons are touching. Then $Z = \frac{1}{2}$.

$$I(Y,\tfrac{1}{2}) + M(Y,\tfrac{1}{2}) = 2\left[-1 + 5(4Y-1)^2/3 - 2Y^2/(1-Y^2)\right.$$

$$-8Y^2/\sqrt{1-Y^4} - 4Y^2/\sqrt{1-Y^2}\sqrt{1+3Y^2} - 16Y(4Y-1)/\sqrt{4-3Y^2}$$

$$+16Y(4Y-1)/\sqrt{9-8Y^2} + 4\sqrt{2}\,Y^2/\sqrt{2-3Y^2+Y^4}$$

$$\left. + (2Y^2 + 3.83Y)/\sqrt{1-Y^2}\,\right] \qquad (26)$$

These terms are worked out in the accompanying table. $I(Y,\frac{1}{2}) + M(Y,\frac{1}{2})$ is plotted in Figure 7. It shows the requisite packing occurs when Y is about .98.

This table was computed with a ten inch slide rule. Three significant figures are given and reliance can be placed only on the first two. Greater accuracy is unnecessary and indeed would be misleading, in view of the approximate nature of the work.

	(1)	(2)	(3)	(4)	(5)	(6)
(1)	Y	.60	.7	.80	.85	.90
(2)	Y^2	.360	.49	.640	.723	.81
(3)	Y^3	.216	,343	.512	.614	.729
(4)	Y^4	.130	.240	.420	.524	.656
(5)	$Y^{3/2}$	.464	.585	.715	.784	.854
(6)	$4Y - 1$	1.40	1.80	2.20	2.40	2.60
(7)	$(4Y - 1)^2$	1.96	3.24	4.84	5.76	6.76
(8)	$1 - Y^2$	.640	0.510	.360	.277	.19
(9)	$(1 - Y^2)^{1/2}$	.800	.715	.600	.526	.436
(10)	$(1 - Y^2)^{3/2}$	.512	.364	.216	.146	.083
(11)	$(1 - Y^2)^2$	.410	.260	.130	.0767	.0361
(12)	$1 - Y^4$	.870	.760	.590	.476	.344
(13)	$(1 - Y^4)^{1/2}$	.933	.871	.768	.690	.586
(14)	$(1 - Y^4)^{3/2}$	.812	.662	.455	.328	.202
(15)	$2 - 3Y^2 + Y^4$	1.05	.770	.490	.355	.226
(16)	$(2 - 3Y^2 + Y^4)^{1/2}$	1.02	.878	.700	.596	.475
(17)	$(2 - 3Y^2 + Y^4)^{3/2}$	1.08	.675	.343	.212	.107
(18)	$4 - 3Y^2$	2.92	2.53	2.08	1.831	1.570
(19)	$(4 - 3Y^2)^{1/2}$	1.71	1.59	1.44	1.35	1.25
(20)	$(4 - 3Y^2)^{3/2}$	4.98	4.03	3.01	2.48	1.97
(21)	$9 - 8Y^2$	6.12	5.08	3.88	3.22	2.52
(22)	$(9 - 8Y^2)^{1/2}$	2.48	2.26	1.97	1.79	1.59
(23)	$(9 - 8Y^2)^{3/2}$	11.5	11.45	7.65	5.77	4.00
(24(	$1 + 3Y^2$	2.08	2.47	2.92	3.17	3.43
(25)	$(1 + 3Y^2)^{1/2}$	1.44	1.57	1.71	1.78	1.58
(26)	$(1 + 3Y^2)^{3/2}$	3.02	3.88	4.99	5.64	6.36

	(7)	(8)	(9)	(10)	(11)	(12)
(1)	.93	.95	.96	.97	.98	.99
(2)	.860	.902	.922	.941	.960	.98
(3)	.804	.857	.895	.913	.942	.97
(4)	.740	.815	.850	.885	.922	.96
(5)	.897	.926	.940	.955	.970	.985
(6)	2.72	2.80	2.84	2.88	2.92	2.96
(7)	7.40	7.84	8.08	8.30	8.53	8.76
(8)	.140	.0975	.0780	.0590	.0400	.0200
(9)	.374	.312	.280	.243	.200	.141
(10)	.0524	.030	.022	.0143	.008	.00282
)11)	.0195	.00951	.0061	.00348	.0016	.000400
(12)	.260	.186	.150	.115	.0780	.0400
(13)	.510	.432	.387	.339	.279	.200
(14)	,133	.080	.058	.0390	.0220	.00800
(15)	.160	.107	.084	.062	.0420	.0200
(16)	.400	.327	.290	.249	.205	.141
(17)	.0640	.035	.0244	.0154	.0086	.00282
(18)	1.42	1.29	1.23	1.18	1.12	1.06
(19)	1.19	1.13	1.11	1.09	1.06	1.03
(20)	1.79	1.47	1.37	1.28	1.18	1.09
(21)	2.12	1.78	1.62	1.47	1.32	1.16
(22)	1.46	1.33	1.27	1.21	1.15	1.08
(23)	3.08	2.37	2.06	1.78	1.52	1.25
(24)	3.58	3.71	3.77	3.82	3.88	3.94
(25)	1.89	1.92	1.94	1.95	1.97	1.98
(26)	6.78	7.14	7.30	7.45	7.64	7.82

	(1)	(2)	(3)	(4)	(5)	(6)
(27)	$5(4Y-1)^2/3$	3.27	5.40	8.08	9.61	11.3
(28)	$-2Y^2/(1-Y^2)$	-1.13	-1.92	-3.56	-5.30	-8.53
(29)	$-8Y^2/\sqrt{1-Y^4}$	-3.08	-4.50	-6.58	-8.40	-11.1
(30)	$-\frac{4Y^2}{\sqrt{1-Y^2}\sqrt{1+3Y^2}}$	-1.25	-1.74	-2.50	-3.09	-4.71
(31)	$-\frac{16Y(4Y-1)}{\sqrt{4-3Y^2}}$	-7.85	-12.7	-19.5	-24.2	-30.0
(32)	$\frac{16Y(4Y-1)}{\sqrt{9-8Y^2}}$	5.42	8.93	14.3	18.3	23.6
(33)	$\frac{4\sqrt{2}\,Y^2}{\sqrt{2-3Y^2+Y^4}}$	1.99	3.15	5.16	6.85	9.61
(34)	$\frac{2Y^2+3.83Y}{\sqrt{1-Y^2}}$	3.78	5.21	7.25	8.95	11.6
(35)	$I(Y,\frac{1}{2})+M(Y,\frac{1}{2})$	.30	1.66	3.30	3.44	1.54
(36)	$Y/\sqrt{2}\sqrt{1-Y^2}$	.53	.69	.95	1.15	1.47
(37)	$Y/4\sqrt{1-Y^2}$	.19	.25	.33	.41	.52
(38)	$4Y(4Y-1)$	3.36	5.04	7.05	8.15	9.35
(39)	$\frac{16Y(4Y-1)(1-Y^2)}{(4-3Y^2)^{3/2}}$	1.72	2.53	3.36	3.64	3.60
(40)	$-2Y^2/(1-Y^2)$	-1.12	-1.92	-3.56	-5.22	-8.54
(41)	$-\frac{2Y^2(1+Y^2)}{\sqrt{1-Y^2}\,(1+3Y^2)^{3/2}}$	-.41	-.53	-.70	-.84	-1.06
(42)	$-\frac{4Y^2}{\sqrt{1-Y^4}}$	-1.54	-2.25	-3.33	-4.19	-5.53
(43)	$R(Y)$	3.55	6.30	13.1	20.8	36.9

	(7)	(8)	(9)	(10)	(11)	(12)
(27)	12.3	13.1	13.5	13.9	14.2	14.5
(28)	-12.3	-18.5	-23.6	-31.9	-48.0	-98.0
(29)	-13.5	-16.7	-19.1	-22.3	-27.6	-39.2
(30)	-4.87	-6.01	-6.79	-7.97	-9.75	-14.0
(31)	-34.0	-37.5	-37.9	-41.1	-43.3	-45.6
(32)	27.8	31.9	34.2	36.9	39.8	43.5
(33)	11.2	15.6	17.9	21.4	26.4	40.0
(34)	14.1	17.4	19.7	23.0	28.4	40.4
(35)	-.44	-3.42	-6.18	-18.14	-41.70	-119.8
(36)	1.77	2.16	2.41	2.48	3.48	
(37)	.623	.763	.86	1.00	1.22	
(38)	10.1	10.6	10.9	11.2	11.4	
(39)	+3.16	+2.83	2.46	2.06	1.54	
(40)	-12.3	-18.5	-23.7	-31.9	-48.0	
(41)	-1.26	-1.54	-1.73	-2.02	-2.46	
(42)	-6.75	-8.35	-9.53	-11.1	-13.8	
(43)	54.9	88.5	116	160	247	

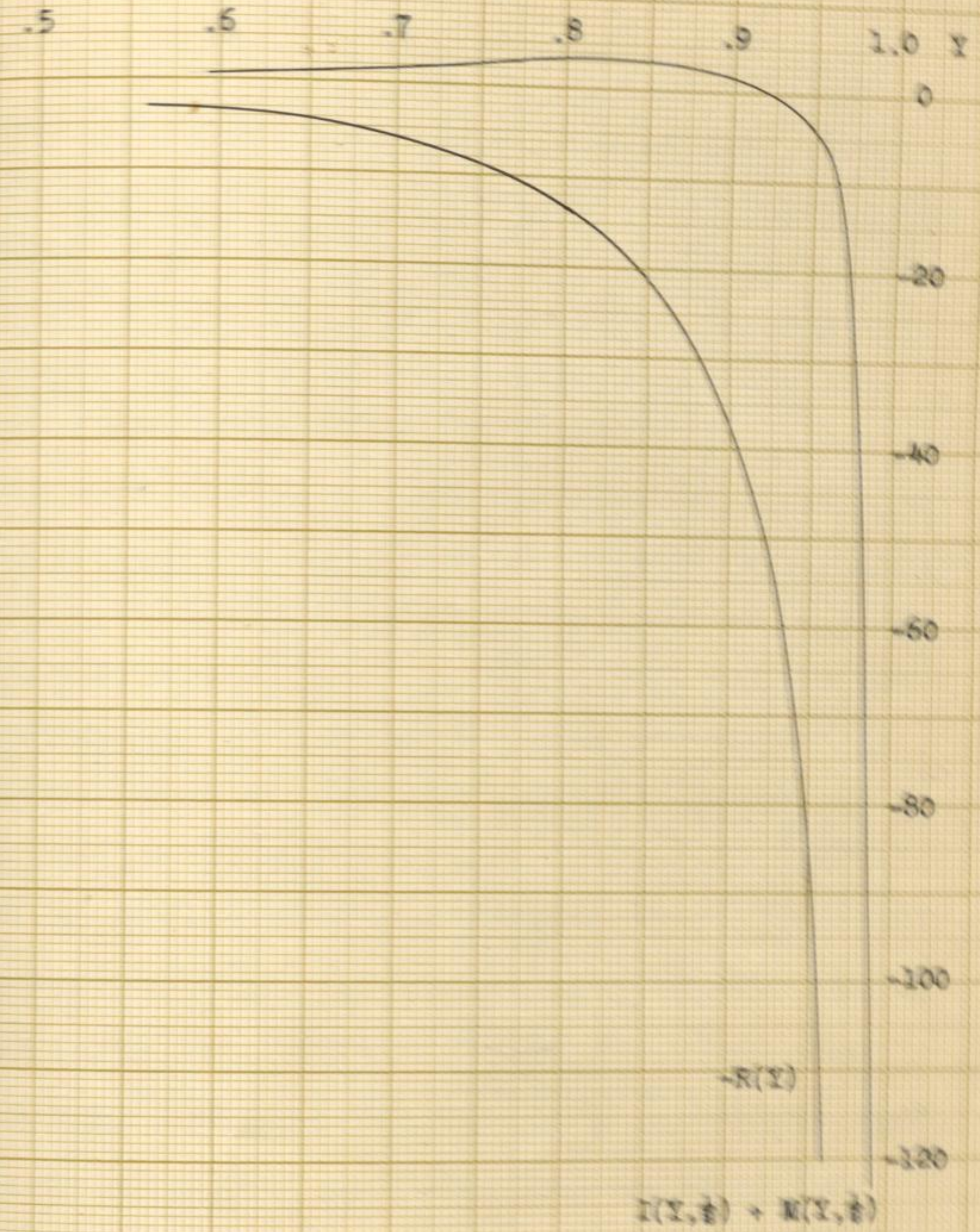


Figure 7

As has been remarked, secondary and further images have been left out of consideration, in order to leave the problem simple enough to be attractive. If the images A''', B''', C''', D''' in electron E of A'', B'', C'', D'', and $A^{iv}, B^{iv}, C^{iv}, D^{iv}$ in F of A', B', C', D' had been included, the value of $I(Y,\frac{1}{2}) + M(Y,\frac{1}{2})$ would have been much smaller, i.e. the mutual mass would have turned out to be considerable more negative, for any value of Y than the results given in the table and graph. The inclusion of A''', B''', C''', D''', $A^{iv}, B^{iv}, C^{iv}, D^{iv}$, they being all positive, would increase the packing in several ways. (i) Their direct effect on the intrinsic mass of each electron would be a decrease, because placing a positive distribution on the negative distributions would give a smaller value of $\iint \frac{de_1 de_2}{r}$. (ii) q_2 would change from $-\frac{(4Y-1)}{1-Z}Z$ to $-\frac{(4Y-1)e-A'''}{1-Z}Z$. This would also decrease the value of $\iint \frac{de_1 de_2}{r}$. (iii) q_1 would change from $\frac{(4Y-1)}{1-Z}$ to $\frac{(4Y-1)e-4A'''}{1-Z}$. This, too, would lower $\iint \frac{de_1 de_2}{r}$ thus decreasing the intrinsic mass. This new value of q_1 would also lower the mutual mass thru the terms $(E,A), \cdots$, $(F,A), \cdots$. (iv) The new terms like (A''',A''), (A''',B''), $\cdots$, (A^{iv},A'), $\cdots$ being 32 in number would lower the mutual mass by nearly offsetting the positive terms (A',A''), (A',B''), $\cdots$, (A'',A'), $\cdots$. A slight disadvantage would arise from terms such as (A,A'''), (A,B'''), ... (A,A^{iv}) ... (D,D^{iv}), but the net effect would be a considerably greater packing effect. Instead of

.98 for Y we should have more likely about .9.

A study of Figure 7 will convince anyone that the first approximation yields values for $I(Y,\frac{1}{2}) + M(Y,\frac{1}{2})$ which are too high. The sum of these two functions can never be positive, yet the curve crosses the Y axis. This shows clearly that a better approximation will lower the I + M curve.

The object of this section is not to find the numerical value of Y which yields the requisite packing, but to prove that the requisite amount <u>can</u> be accounted for. The object has been accomplished, even under the adverse condition of excluding all but primary images. Indeed even more than the required -57m is obtained by making Y near enough to unity. With closer approximations, we would probably find that Y could be adjusted to decrease the atomic weight perhaps a half; this is only a hazardous guess, of course.

The force on electron E, in the direction of FE, is

$$\begin{aligned}\frac{1}{4\pi}\Big[\, & q_1^2/\overline{EF}^2 + 2q_1 q_2/\overline{EF''}^2 + 4q_1 q_3 a(2-Y^2)/\overline{EA''}^3 \\ & + 4\Big\{ q_3^2/\overline{A'A''}^2 + q_3^2\,\overline{A'A''}/\overline{A'C''}^3 + 2q_3^2\,\overline{A'A''}/\overline{A'B''}^3 \\ & + q_1 q_3 a(2-Y^2)/\overline{EA''}^3 + 2q_1 q_2 a(1.5 - Y^2)/\overline{A'F''}^3 \\ & + q_2^2/\overline{F'F''}^2 + 4\Big\{ eq_1 Y/\overline{AE}^2 + eq_3\, \tfrac{1}{2}\overline{A'A''}/\overline{AA'}^3 \\ & + eq_3\, \tfrac{1}{2}\overline{A'A''}/\overline{AC'}^3 + 2eq_3\, \tfrac{1}{2}\overline{A'A''}/\overline{AB'}^3 + eq_2\, \tfrac{1}{2}a/\overline{AF'}^3 \Big\}\Big] \qquad (27)\end{aligned}$$

These terms are respectively the forces due to charges at E and F, E and F", E and B", C' and C", C' and A", C' and B", C' and F, C' and F", F' and F", C and E, C and C', C and A', C and B", C and F'. The coefficient 2 in the second term arises because the force between E and F" is exactly the same as between F' and C. The remaining numerical coefficients are similarly explained. This becomes

$$\frac{e^2}{4\pi a^2}\Big[(4Y-1)^2 - 4(4Y-1)^2/2.25 - 8Y(4Y-1)(2-Y^2)/(4-3Y^2)^{3/2}$$

$$+ 4\left\{ Y^2/4(1-Y^2)^2 + Y^2/4\sqrt{1-Y^2} + \sqrt{2}\, Y^2(1-Y^2)/(2-3Y^2+Y^4)^{3/2} \right.$$
$$- 2Y(4Y-1)(2-Y^2)/(4-3Y^2)^{3/2} - 32(4Y-1)^2(1.5-Y^2)/(9-8Y^2)^{3/2}\Big\}$$
$$+ (4Y-1)^2/4(1-Y^2)^2 + 4\Big\{2Y^3(4Y-1) - Y^4/(1-Y^2)^2$$
$$- Y^4/\sqrt{1-Y^2}\,(1+3Y^2)^{3/2} - 2Y^4(1-Y^2)/(1-Y^4)^{3/2}$$
$$\left. - 4Y^3(4Y-1)/(4-3Y^2)^{3/2}\Big\}\right]$$

$$= \frac{e^2}{4\pi a^2}\Big[-.778(4Y-1) - \frac{32Y(4Y-1)}{(4-3Y^2)^{3/2}} - \frac{16Y^4-(4Y-1)^2-4Y^2}{4(1-Y^2)^2}$$
$$+ \frac{Y^2}{\sqrt{1-Y^2}} + 4\sqrt{2}\,Y^2(1-Y^2)/(2-3Y^2+Y^4)^{3/2}$$
$$- 128(4Y-1)^2(1.5-Y^2)/(9-8Y^2)^{3/2} + 8Y^3(4Y-1)$$
$$- 4Y^4/\sqrt{1-Y^2}\,(1+3Y^2)^{3/2} - 8Y^4(1-Y^2)/(1-Y^4)^{3/2}\Big] \qquad (28)$$

The coefficient of $e^2/4\pi a^2$ is negative, at least when $Y > .6$. As Y approaches unity, it becomes relatively large in magnitude. This means that the electrons are being pulled together with considerable force. So the Lenz model would be very stable, which is just what is required by the known facts concerning the stability of α particles.

Here, and in all other cases where the force on electrons and protons is desired, the ordinary electrostatic interaction will suffice. The protons are not moving so fast that it is necessary to use the series expression for the simultaneous field of a moving charge. The introduction of any such refinement would be quite out of place unless the whole problem is carried to a much closer approximation. Also, the series expression would show the orbits to be unstable on account of their radiation of energy.

COMPLICATIONS ARISING FROM DYNAMICAL EQUILIBRIUM

The protons must be revolving in order to be in dynamic equilibrium. The necessary peripheral speed βc increases the space average mass of the four protons by the amount

$$4 \times 1846\ m\left(\frac{2k+k^3}{3} - 1\right) \doteq 4 \times 1846\ m\ \left(\frac{5}{6}\beta^2\right)$$

if $\beta^2 << 1$. k denotes $(1-\beta^2)^{-\frac{1}{2}}$. β will now be found by equating the radial electrostatic force on a proton to its kinetic reaction in rotation.

In Figure 6, if $Z = \frac{1}{2}$, then

Radius of rotation $OA = aY^{-1}\sqrt{1-Y^2}$

$AC = 2aY^{-1}\sqrt{1-Y^2}$ $\quad$ $AB = \sqrt{2}\ aY^{-1}\sqrt{1-Y^2}$

$EF = 2a$ $\quad$ $EA = a/Y$

$A'A'' = 2a(1-Y^2)$ $\quad$ $A'C'' = 2a\sqrt{1-Y^2}$

$A'B'' = \sqrt{2}\ a\ \sqrt{2-3Y^2+Y^4}$ $\quad$ $AB' = aY^{-1}\sqrt{1-Y^4}$

$AA' = aY^{-1}(1-Y^2)$ $\quad$ $AF' = \frac{1}{2}aY^{-1}\sqrt{4-3Y^2}$

$AC' = aY^{-1}\sqrt{1-Y^2}\ \sqrt{1+3Y^2}$ $\quad$ $EC'' = a\sqrt{4-3Y^2}$

$EF' = \frac{1}{2}a$ $\quad$ $q_1 = 2(4Y-1)e$

$q_2 = -(4Y-1)e$ $\quad$ $q_3 = -eY$

The electrostatic force acting on A, resolved along the radius, due to B, C, E, F', A', C', B', in order will now be written. The coefficient (2) arises when two forces are exactly alike, for example those due to B and D.

$$
\begin{aligned}
&-\frac{1}{4\pi a^2}\Big\{(2)e^2Y^2/2\sqrt{2}\,(1-Y^2) + e^2Y^2/4(1-Y^2) \\
&+ (2)\Big[\,eq_1 Y^2\sqrt{1-Y^2} + eq_2\, 8Y^2\sqrt{1-Y^2}/(4-3Y^2)^{3/2} + eq_3 Y^2/(1-Y^2)^{3/2} \\
&+ eq_3 Y^2(1+Y^2)/(1-Y^2)(1+3Y^2)^{3/2} + (2)eq_3 Y^2(1+Y^2)\sqrt{1-Y^2}/(1-Y^4)^{3/2}\Big]\Big\} \\
&= -\frac{e^2}{4\pi a^2}\Big\{Y^2/\sqrt{2}\,(1-Y^2) + Y^2/4(1-Y^2) + 4(4Y-1)\,Y^2\sqrt{1-Y^2} \\
&\quad - 16(4Y-1)\,Y^2\sqrt{1-Y^2}/(4-3Y^2)^{3/2} - 2Y^3/(1-Y^2)^{3/2} \\
&\quad - 2Y^3(1+Y^2)/(1-Y^2)(1+3Y^2)^{3/2} - 4Y^3(1+Y^2)\sqrt{1-Y^2}\,/(1-Y^4)^{3/2}\Big\} \\
&= \frac{e^2}{4\pi a^2}\,P(Y) \text{ say.}
\end{aligned} \tag{29}
$$

This must balance the kinetic reaction

$$
M\,v^2/r\sqrt{1-\beta^2} = \frac{e^2}{4\pi a^2}\cdot\frac{1846 \times 2}{3}\cdot\frac{Y}{\sqrt{1-Y^2}}\cdot\frac{\beta^2}{\sqrt{1-\beta^2}} \tag{30}
$$

i.e. $\dfrac{1846 \times 2}{3}\cdot\dfrac{Y}{\sqrt{1-Y^2}}\cdot\dfrac{\beta^2}{\sqrt{1-\beta^2}} = P(Y)$

$$
\frac{\beta^2}{\sqrt{1-\beta^2}} = \frac{3}{1846 \times 2}\;\frac{\sqrt{1-Y^2}}{Y}\,P = p \text{ say}
$$

$$
\beta^2 = p \text{ approximately.} \tag{31}
$$

The relativity increase in mass of the four protons is

$$
4 \times 1846\;\; m\,(5p/6) = \frac{10 \times 1846}{3}\,p\,m = m\,R(Y) \text{ say.}
$$

$$
\begin{aligned}
R(Y) &= 5\,\frac{\sqrt{1-Y^2}}{Y}\,P(Y) = -5\Big\{Y/\sqrt{2}\,\sqrt{1-Y^2} + Y/4\sqrt{1-Y^2} \\
&+ 4(4Y-1)Y - 16(4Y-1)\,Y(1-Y^2)/(4-3Y^2)^{3/2} - 2Y^2/(1-Y^2) \\
&- 2Y^2(1+Y^2)/(1-Y^2)^{1/2}(1+3Y^2)^{3/2} - 4Y^2/\sqrt{1-Y^4}\Big\}
\end{aligned} \tag{32}
$$

This is plotted in Figure 7 along with $I(Y,\frac{1}{2}) + M(Y,\frac{1}{2})$. It amounts to more than the packing effect. This indicates that, with the present stage of approximation, the gain in negative mutual mass is more than offset by the relativity increase.

A closer approximation would effect a decrease in R(Y) and, as explained previously, an algebraic decrease in $I(Y,\frac{1}{2}) + M(Y,\frac{1}{2})$. It is possible that these two effects, operating as they do in the same direction, might go so far as to allow the required packing along with dynamical equilibrium of the protons. The writer hopes to settle this question in future work. It must be admitted that the outlook is not very bright.

It should be noticed that R(Y) is not dependent on the number 1846, because it cancelled from numerator and denominator; a:A or M:m could be used in its place.

To mR(Y) there should be added the increase in mass due to the rotating surface distributions on the electrons, but this is small compared to mR(Y) and so is neglected.

The model of Figure 1 is even worse when an attempt is made to put its protons in equilibrium by rotation, because there the force is directed entirely along the radius. In the Lenz model the greatest force on proton A for example is the attraction of its own images, and this attraction is directed at an angle of arc cos Y with the radius, which may be nearly 90°. Hence the relativity increase will be comparatively greater for the first model than for the Lenz model. Since the results obtained for Figure 1 are exact, there is no hope at all for this model, if dynamic equilibrium is necessary.

One consolation is the utter failure of ordinary Lorentz electrons. It will be recalled that with them the mutual mass for the model of Figure 1 was found to be -6 Ym. If the protons are ~~rotating~~ revolving, the relativity increase in mass is $\frac{15}{4}$ Ym. The net decrease is only $\frac{9}{4}$ Ym = 2.25 Ym, which is practically 2.25 m as a maximum. This amount is about the experimental error in atomic weights, being only 2.25:4 x 1847 or 1 part in 3900.

Lorentz electrons in the Lenz model show some interesting features. The details are given below. In Figure 6, charges -e are at E and F, +e at A, B, C, D. $z = \frac{1}{2}$. The mutual mass is

$$-\frac{e^2}{6\pi ac^2}\left\{16Y - 4\,\frac{Y}{2\sqrt{1-Y^2}} - 8\,\frac{\sqrt{2}\,Y}{2\sqrt{1-Y^2}} - 1\right\}$$

$$= m\left\{+1 - 16Y + (2 + 4\sqrt{2})\,Y/\sqrt{1-Y^2}\right\} = m\,G_1(Y) \text{ say.} \qquad (33)$$

The radial force on any proton is

$$\frac{e^2}{4\pi a^2}\left\{2Y^2\sqrt{1-Y^2} - Y^2/4(1-Y^2) - \sqrt{2}\,Y^2/2(1-Y^2)\right\}$$

$= \frac{e^2}{4\pi a^2} P_1(Y)$ say, which must balance the kinetic reaction,

$$\frac{1846\ m}{\sqrt{1-\beta^2}}\cdot\frac{v^2}{r} = \frac{e^2}{4\pi a^2}\cdot\frac{1846 \times 2}{3}\cdot\frac{Y}{\sqrt{1-Y^2}}\cdot\frac{\beta^2}{\sqrt{1-\beta^2}} \qquad (34)$$

So $\dfrac{\beta^2}{\sqrt{1-\beta^2}} = \dfrac{8Y(1-Y^2)^{3/2} - Y - 2\sqrt{2}\,Y}{4\sqrt{1-Y^2}} \times \dfrac{3}{1846 \times 2} = p_1$ say.

$\beta^2 = p_1$ nearly. (35)

The relativity increase in mass of the four protons is as before

$$4 \times 1846\, m(5\beta^2/6) = \frac{10 \times 1846}{3}\, p\, m$$

$$= \frac{8Y(1-Y^2)^{3/2} - Y - 2\sqrt{2}\,Y}{4\sqrt{1-Y^2}}\, 5m \qquad (36)$$

The net packing effect, Mutual mass + Relativity increase, is

$$m\left\{1 - 16Y + 10Y - 10Y^3 + \frac{6\sqrt{2} + 3}{4\sqrt{1-Y^2}}\, Y\right\} = m\, N(Y) \text{ say.} \qquad (37)$$

$P_1(.623) = 0$, which means that the protons are just in equilibrium (unstable of course, by Earnshaw's theorem) with no rotation. The upper limit to Y is thus .623. When Y = .397 the electrons just start to separate. So Y is limited to the interval .397 to .623.

N(Y) and $G_1(Y)$ are plotted in Figure 8. The graph shows the maximum net packing with dynamical equilibrium is -2.87 m using Lorentz electrons in the Lenz model. This again is just about undetectable. One would expect the mutual mass $mG_1(Y)$ to pass thru a minimum as Y varies from 1 to 0, because when Y is nearly unity the protons are very close together. By equating the first derivative to zero it is easily found that Y = .623 for a minimum. This is exactly the position of equilibrium. $G_1(.623)$ is the minimum value of $G_1(Y)$, and $P_1(.623) = 0$.

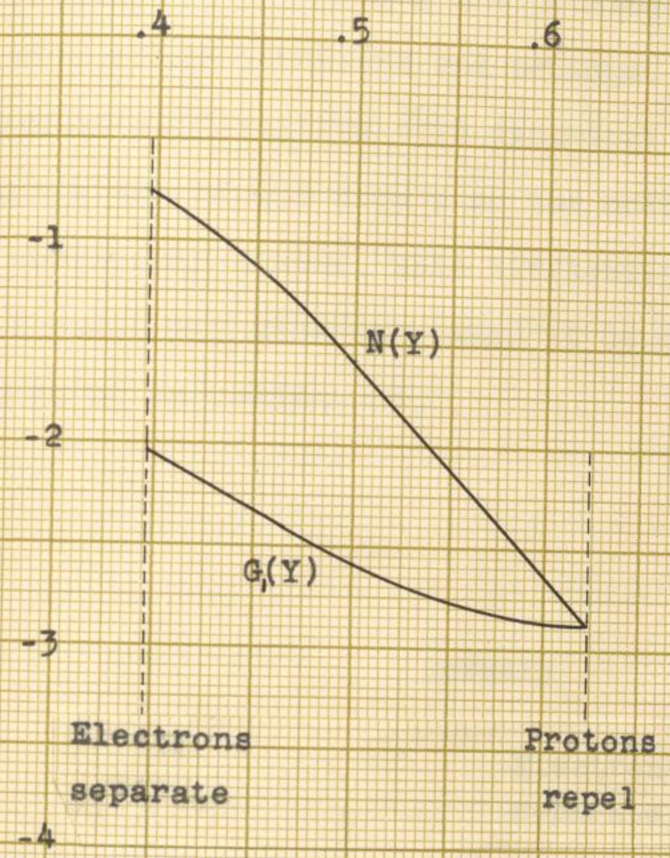


Figure 8

The idea of equipotential electrons is highly successful in explaining the packing effect when applied to stationary electrons and protons, but apparently encounters difficulties with dynamical equilibrium. This trouble may be due solely to the model used for the test. As previously stated, this difficulty in the case of the Lenz model may disappear when it is carried thru a closer degree of approximation.

SUMMARY

The difference in mass between 4H and He is roughly 57 m, m being the rest mass of one electron. Only about one-tenth of this discrepancy in atomic weights is explained by the negative mutual mass of Lorentz electrons and protons (or other electrons having radial law of density) arranged as in Figure 1, a model which allows the electrons to be as close to the protons as in any model imaginable. By postulating that the electron and protons of this static model shall be equipotential surface distributions, it is found that the possible space average mutual mass is decreased (algebraically) and the space average intrinsic mass increased, the algebraic sum of the two being a net decrease in mass m F(Y), which not

only can account for the -57m, but can also reduce the atomic weight more than half, by adjusting $Y \equiv a:f$.

On account of the small radius of the protons, their mass is not sensibly affected by the modification. They can therefore be treated as point charges, as usual, for electrical effects external to them.

The static Lenz model is considerably more complicated. An exact value for Y is not attempted, but an approximate calculation shows the requisite -57 m to be easily rendered. It is proved that a closer approximation can not fail to yield still a greater packing effect than the first approximation, for any given value of Y.

The results obtained with equipotential electrons in the two static models are only specific applications of a modification which can not fail to yield greater packing than ordinary electrons, because of a fundamental law of electrical energy, namely, "If any number of surfaces are fixed in position, and a given charge is placed on each surface, then the energy is a minimum when the charges are placed so that every surface is an equipotential."[3]

When applied to a dynamical model, the modification encounters disaster in the case of the Figure 1 model because of the relativity increase in mass caused by the motion necessary for dynamical equilibrium, and it encounters difficulties for the same reason in the Lenz model, altho it has not been definitely proved a failure there.

The net packing effect (negative mutual mass plus relativity increase) in the two dynamical models using Lorentz electrons (or any electrons with a radial law of density) is easily calculated and is found to be about equal to the experimental error in determining atomic weights. Hence ordinary electrons are total failures in the dynamical models considered.

REFERENCES

1. W. D. Harkins and E. D. Wilson, J. Amer. Chem. Soc. 37, pp. 1367-1383, June 1915.
The problem of mutual electromagnetic momentum and energy for two charges (and hence for any number of charges) constituting a uniformly convected system has been solved by Irwin Roman, Proc. Nat. Acad. Sci., 9, p. 165. (1923)
2. See, for example, Leigh Page, "An Introduction to Electrodynamics", Ginn & Company, 1922, equation 33 page 56.
3. Jeans, "Electricity and Magnetism", Cambridge University Press, 1923, article 189.
4. See, for example, Leigh Page, ibid., p. 52.
5. Remembering that $\int_{-1}^{+1} P_n P_m d\mu = \begin{cases} 2/(2n+1) & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$
6. W. Lenz, Münchener Akademie 1918, S. 355. Also described in Sommerfeld's Atombau und Spektrallinien, 3. Auflage, S. 118.
7. J. C. Maxwell, "Electricity & Magnetism", Vol. I, Ch. XI.

W. Lenz, Sitzungsberichte der mathematisch-physikalischen Klasse der Bayerischen Akademie der Wissenschaften zu München, (1918), S. 355. etc.